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Fixed Point Theorems in Fuzzy 2 – Banach Space Using CLR Property and Implicit Relation

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Abstract:
The purpose of this paper is to obtain common fixed point theorems for weakly compatible mappings satisfying the Common Limit Range property using implicit relation in Fuzzy 2 – Banach Space. This Property plays a major role in fixed point theorems and by using this property we can obtain fixed points in Fuzzy 2-Banach Space.
Key Words: *Common fixed point; Weakly Compatible Maps; Fuzzy 2-Banach Space; Property CLR.*

1. Introduction

The concept of Fuzzy Sets was introduced by Zadeh in 1965, which plays a major role in almost all branches of Science and Engineering. Katsaras (1984) and Congxin and Ginxuan (1984) independently introduced the definition of fuzzy norms. The concept of 2 – norm in linear spaces was initiated by Gahler (1964) and white (1969) introduced the concept of Cauchy sequences and convergent sequences in a 2-normed spaces and defined a 2- Banach Spaces as a 2-normed space in which every Cauchy sequence is convergent. Sintunawarat and kumam introduced a new concept called as common limit range property.

2. Preliminaries

Definition 2.1. [4]

Let D be a vector space over a field K (where K is R or C) and * be a continuous t-norm. A fuzzy set N in $D^2X[0,\infty]$ is called a fuzzy 2-norm on D if it satisfies the following conditions:

(i) $N(p,q,0)=0\ \forall\ p,q\in D$

(ii) $N(p,q,t)=1\ \forall t>0$ and atleast two among the three points are equal

(iii) $N(p,q,t)=N(q,p,t)$

(iv) $N(p+q+r,t_1+t_2+t_3)\geq N(p,q,t_1)*N(p,r,t_2)*N(q,r,t_3)\forall p,q,r\in S$ and $t_1,t_2,t_3\geq 0$

(v) For every $p,q,\in D, N(p,q,.)$ is left continuous and $\lim_{t\to\infty} N(p,q,t)=1$

The triple $(D,N,*)$ will be called fuzzy 2-normed linear space $(F2-NLS)$

Definition 2.2.[4]

A sequence $\{P_n\}$ in a $F2-NLS(D,N,*)$ is converge to $p\in D$ if and only if

$\lim_{n\to\infty} N(P_n,p,t)=1\forall t>0$

Definition 2.3. [4]

Let $(D,N,*)$ be a $F2-NLS$. A sequence $\{P_n\}$ in D is called a fuzzy Cauchy sequence if and only if

$\lim_{m,n\to\infty} N(P_m,P_n,t)=1\forall t>0$

Definition 2.4. [4]

A linear fuzzy 2-normed space which is complete is called a fuzzy 2 – Banach Space.

Definition 2.5.[4]

Self mappings A and S of a fuzzy 2- Banach Space $(D, N,*)$ are said to be weakly commuting if $N(ASp, SAp, t) \geq N(Ap, Sp, t) \forall p \in D$ and $t > 0$.

Definition 2.6.[4]

Self mapping A and S of a fuzzy 2 – Banach Space $(D, N,*)$ are said to be compatible if and only if $\lim_{n\to\infty}(ASp_n, SAp_n, t) = 1 \forall t > 0$

. Whenever $\{P_n\}$ is a sequence in D such that $Ap_n, Sp_n \to p$ for some $p \in D$ as $n \to \infty$.

Definition 2.7.[4]

Two Self maps A and S are said to be commuting if $ASp = SAp$ for all $p \in D$.

Definition 2.8.[4]

Let A and S be two self maps on a set D, if $Ap = Sp$ for some $p \in D$ then p is called a coincidence point of A and S.

Definition 2.9.[4]

Two Self maps A and S of a fuzzy 2 – Banach Space $(D, N,*)$ are said to be weakly compatible if they commute at their coincidence points. That is if $Ap = Sp$ for some $p \in D$ then $ASp = SAp$.

Definition 2.10.[4]

Suppose A and S be two Self mappings of fuzzy 2-Banach Space $(D, N *)$. A point $p \in D$ is called a coincidence point of A and S if and only if $Ap = Sp$, then $w = Ap = Sp$ is called a point of Coincidence of A and S.

Definition 2.11.[1]

A pair (A, S) of Self mapping of a fuzzy 2 – Banach Space $(D, N,*)$ is said to satisfy property $(E.A)$ if there exists a sequence $\{P_n\}$ in D such that $\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = z$

for some $z \in D$.

Definition 2.12.[1]

Two pairs (A, S) and (B, T) of a self mappings of a fuzzy 2-Banach Space $(D, N,*)$ are aid to satisfy the common property $(E.A)$ if there exist two sequence $\{p_n\}, \{q_n\}$ in D. Such that $\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Bq_n = \lim_{n\to\infty} Tq_n = z$

for some $z \in D$.

3. Implicit relations : [5]

Let $\{\emptyset\}$ be the set of all real continuous function $\emptyset: (R^+)^6 \to R^+$ satisfying the following condition:

(i) $\emptyset(u, v, u, v, v, u) \geq 0$ imply $u \geq v$ for all $u, v \in [0,1]$

(ii) $\emptyset(u, v, v, u, u, v) \geq 0$ imply $u \geq v$ for all $u, v \in [0,1]$

(iii) $\emptyset(u, u, v, v, u, u) \geq 0$ imply $u \geq v$ for all $u, v \in [0,1]$

Lemma 3.1.[6]

Let $(D, N,*)$ be a fuzzy 2-Banach Space. If there exists $k \in (0,1)$ Such that $N(p, q, kt) \geq N(p, q, t)$ for all $p, q \in D$ and $t > 0$ then $p = q$.

Lemma 3.2.

Two Self mapping A and S of a fuzzy 2-Banach Space $(D, N,*)$ are compatible then (A, S) is weakly compatible

Lemma 3.3.

Two Self mapping A and S of a fuzzy 2-Banach Space $(D, N,*)$. (A, S) is weakly compatible. w is a point of coincidence of A and S then the pair (A, S) satisfies the property *CLR*.

Theorem 3.4.

Two Self mapping A and S of a fuzzy 2-Banach Space $(D, N,*)$.

(i) Pair (A, S) is weakly compatible.

(ii) Pair (A, S) is compatible.

(iii) w is a point of coincidence of A and S.

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(iv) Pair (A, S) satisfies the property CLR then A and S have a fixed point.

Theorem 3.5

Let $(D, N, *)$ be a fuzzy 2-Banach space and $K, L, M, T, R, S : D \to D$ be hexadic self-mappings satisfy the following conditions:

(i) one of the pairs (K, TR) and (L, MS) satisfies the property CLR with respect to mapping TR and MS such that $K(D) \subseteq MS(D)$ and $L(D) \subseteq TR(D)$.

(ii) For every $p, q \in D$ and for some $\phi \in (\phi)$ and every $t > 0$,

$\phi\{N(Kp, Lq, t), N(TRp, MSq, t), N(Lq, MSq, t), N(Kp, TRp, t), N(Kp, MSq, t), N(TRp, Lq, t)\} \geq 0$

(iii) If one of $MS(D)$ and $TR(D)$ are closed subset of D.

(iv) Pairs (K, TR) and (L, MS) are weakly compatible.

(v) Each pair of pairs (K, TR) and (L, MS) has a coincidence point in D.

(iv) If $(K, S), (L, R), (MS, R)$ and (TR, S) are commuting pairs than K, L, M, T, R, S have a unique common fixed point in D.

Proof:

Let pairs (K, TR) satisfy CLR property, so there exists a sequence $\{p_n\}$ in D such that $\lim_{n\to\infty} Kp_n = \lim_{n\to\infty} TRp_n = z$ for some z belongs to TR(D).

Since $K(D) \subseteq MS(D)$, there exists $\{q_n\}$ in D such that $Kp_n = MSq_n$ and we get

$\lim_{n\to\infty} Kp_n = \lim_{n\to\infty} TRp_n = \lim_{n\to\infty} MSq_n = TR(z) = z$

We claim that $\lim_{n\to\infty} Lq_n = z$

Put $p = p_n$ and $q = q_n$.

$\phi\{N(Kp_n, Lq_n, t), N(TRp_n, MSq_n, t), N(Lq_n, MSq_n, t), N(Kp_n, TRp_n, t), N(Kp_n, MSq_n, t),$
$N(TRp_n, Lq_n, t)\} \geq 0$

$\phi\{N(z, Lq_n, t), N(z, z, t), N(Lq_n, z, t), N(z, z, t), N(z, z, t), N(z, Lq_n, t)\} \geq 0$

$\phi\{N(Lq_n, z, t), N(z, z, t), N(Lq_n, z, t), N(z, z, t), N(z, z, t), N(Lq_n, z, t)\} \geq 0$

$\phi\{N(Lq_n, z, t), 1, N(Lq_n, z, t), 1,1, N(Lq_n, z, t)\} \geq 0$

$\lim_{n\to\infty} Lq_n = z \qquad [\phi\ (u, v, u, v, v, u) \geq 0\ \Rightarrow u \geq v]$

$\lim_{n\to\infty} Kp_n = \lim_{n\to\infty} TRp_n = \lim_{n\to\infty} MSq_n = \lim_{n\to\infty} Lq_n = z$

Since $MS(D)$ is a closed subset of D, there exists $u \in D$ such that $MSu = z$ and w get

$\lim_{n\to\infty} Kp_n = \lim_{n\to\infty} TRp_n = \lim_{n\to\infty} MSq_n = \lim_{n\to\infty} Lq_n = MSu$

We assert that $Lu = z$

Put $p = p_n$ and $q = u$

$\phi\{N(Lu, MSu, t), N(Kp_n, TRp_n, t), N(Kp_n, Lu, t), N(TRp_n, MSu, t), N(Kp_n, MSu, t),\ N(TRp_n, Lu, t)\} \geq 0$

$\phi\{N(Lu, z, t), N(z, z, t), N(Lu, z, t), N(z, z, t), N(z, z, t),\ N(Lu, z, t)\} \geq 0$

$\therefore Lu = z \quad [\because \phi(u, v, u, v, v, u) \geq 0 \Rightarrow u \geq v$

Thus $Lu = z$ and $MSu = z$

$\therefore Lu = MSu = z$

Since $L(D) \subseteq TR(D)$, there exists $v \in D$ such that $Lu = TRv$ and we get that $Lu = MSu = TRv = z$

To Prove $Kv = z$

Put $p = v$ and $q = u$ $\phi\{N(Kv, Lu, t), N(TRv, MSu, t), N(Lu, MSu, t), N(Kv, TRv, t), N(Kv, MSu, t),$
$N(TRv, Lu, t)\} \geq 0$

$\phi\{N(Kv, z, t), N(z, z, t), N(z, z, t), N(Kv, z, t), N(Kv, z, t), N(z, z, t)\} \geq 0$

$\therefore Kv = z$

Thus $Kv = z$ and $TRv = z$

$\therefore Kv = TRv = z$

We get $Lu = MSu = TRv = Kv = z$

Let (K, TR) and (L, MS) be weakly compatible pairs.

We have, $Kv = TRv \Rightarrow TRKv = LMSv$

$\Rightarrow TRz = Kz$

$\Rightarrow Kz = TRz$

Also, $Lu = MSu \Rightarrow MSLu = LMSu$

$\Rightarrow MSz = lz$

$\Rightarrow Lz = MSz$

Hence z is the coincidence point of each pair (K, TR) and (L, MS).

Next we have to show that z is the common fixed point of K, L, M, T, R and S.

For this, we claim that $Kz = z$

Put $p = z, q = u$

$\phi\{N(Kz, Lu, t), N(TRz, MSu, t), N(Lu, MSu, t), N(Kz, TRz, t), N(Kz, MSu, t), N(TRz, Lu, t)\} \geq 0$

$\phi\{N(Kz, z, t), N(Kz, z, t), N(z, z, t), N(Kz, Kz, t), N(Kz, z, t), N(Kz, z, t)\} \geq 0$

$\phi\{N(Kz, z, t), N(Kz, z, t), 1,1, N(Kz, z, t), N(Kz, z, t)\} \geq 0$

$\therefore Kz = z\ [\because \phi(u, u, v, v, u, u) \geq 0 \Rightarrow u \geq v]$

$\therefore TRz = z\ [\because Kz = TRz]$

$\therefore TRz = Kz = TRz$

We claim that $Lz = z$

Put $p = v, q = z$.

$\phi\{N(Kz, Lz, t), N(TRv, MSz, t), N(Lz, MSz, t), N(Kv, TRv, t), N(Kv, MSz, t), N(TRv, MSz, t)\} \geq 0$

$\phi\{N(Lz, z, t), N(Lz, z, t), N(Lz, Lz, t), N(z, z, t), N(z, Lz, t), N(z, Lz, t)\} \geq 0$

$\phi\{N(Lz, z, t), N(Lz, z, t), 1,1, N(z, Lz, t), N(z, Lz, t)\} \geq 0$

$\therefore Lz = z\ [V(u, u, v, v, u, u) \geq 0 \Rightarrow u \geq v]$

$Lz = MSz = z$

$\because Kz = MSz = Lz = TRz = z$

Since (K, S) and (TR, S) are commuting pairs, we get $K(Sz) = S(Kz) = Sz$.

Also $TR(Sz) = S(TRz) = Sz$.

From here it follows that , $L(Rz) = TR(Sz) = Sz$.

Since (L, R) and (MS, R) are commuting pairs we have,

$L(Rz) = R(Lz) = Rz$ and $MS(Rz) = R(MSz) = Rz$;

From here it follows that , $L(Rz) = MS(Rz) = Rz$

Put $P = Sz$ and $q = z$

$\phi\{N(KSz, Lz, t), N(TRSz, MSz, t), N(Lz, MSz, t), N(KSz, TRSz, t), N(KSz, MSz, t),\ N(TRSz, Lz, t)\} \geq 0$

$\phi\{N(Sz, z, t), N(Sz, z, t), N(z, z, t), N(Sz, Sz, t), N(Sz, z, t),\ N(Sz, z, t)\} \geq 0$

$\phi\{N(Sz, z, t), N(Sz, z, t), 1,1, N(Sz, z, t),\ N(Sz, z, t)\} \geq 0$

$\therefore Sz = z\ [\because \phi(u, u, v, v, u, u) \geq 0 \Rightarrow u \geq v]$

Now $MSz = z$

$\Rightarrow Mz = z\ [\because Sz = z]$

$\Rightarrow Kz = Lz = Mz = Sz = TRz = z$

Put $p = z$ and $q = Rz$

$\phi\{N(Kz, LRz, t), N(TRz, MSRz, t), N(LRz, MSRz, t), N(Kz, TRz, t), N(Kz, MSRz, t),\ N(TRz, MSRz, t)\} \geq 0$

$\phi\{N(z, Rz, t), N(z, Rz, t), N(Rz, Rz, t), N(z, z, t), N(z, Rz, t),\ N(z, Rz, t)\} \geq 0$

$\phi\{N(Rz, z, t), N(Rz, z, t), 1,1, N(Rz, z, t),\ N(Rz, z, t)\} \geq 0$

$\therefore Rz = z$

Also $Tz = z$ as $TRz = z$

$\therefore Kz = Lz = Mz = Tz = Rz = Sz = z$

z is the common fixed point of K, L, M, T, R and S in D.

Similarly if (L, MS) satisfies property CLR and $TR(D)$ is closed subset of D, then we prove that z is a common fixed point of K, L, M, T, R and S in Din the same argument as above.

Uniqueness:

w is also a common fixed point in D.

Put $p = z$ and $q = w$

$\phi\{N(Kz, Lw, t), N(TRz, MSw, t), N(Lw, MSw, t), N(Kz, TRz, t), N(Kz, MSw, t), \ N(TRz, Lw, t)\} \geq 0$

$\phi\{N(z, w, t), N(z, w, t), N(w, w, t), N(z, z, t), N(z, w, t), \ N(z, w, t)\} \geq 0$

$\phi\{N(z, w, t), N(z, w, t), 1,1, N(z, w, t), \ N(z, w, t)\} \geq 0$

$\therefore z = w$ $[\because \phi(u, u, v, v, u, u) \geq 0 \ \Rightarrow u \geq v]$

Hence z is the unique common fixed point of K, L, M, T, R and S in D respectively.

Theorem 3.6

Let $(D, N, *)$ be a fuzzy 2-Banach space and $K, L, M, T, R, S : D \to D$ be six self-mappings satisfy the following conditions:

(i) If (K, TR) satisfies the property CLR such that $K(D) \subseteq MS(D)$ and $L(D) \subseteq TR(D)$.

(ii) For some $\phi \in (\phi)$ and for every $p, q \in D$ and every $t > 0$,

$\phi\{N(Kp, Lq, t), N(TRp, MSq, t), N(Lq, MSq, t), N(Kp, TRp, t), N(Kp, MSq, t), N(TRp, Lq, t)\} \geq 0$

(iii) If $MS(D)$ is a closed subset of D.

(iv) Pairs (K, TR) and (L, MS) are weakly compatible.

(v) Each pair of pairs (K, TR) and (L, MS) has a coincidence point in D.

(iv) If $(K, S), (L, R), (MS, R)$ and (TR, S) are commuting pairs than K, L, M, T, R, S have a unique common fixed point in D.

Theorem 3.7

Let $(D, N, *)$ be a fuzzy 2-Banach space and $K, L, M, T, R, S : D \to D$ be six self-mappings satisfy the following conditions:

(i) If (L, MS) satisfies the property CLR such that $K(D) \subseteq MS(D)$ and $L(D) \subseteq TR(D)$.

(ii) For some $\phi \in (\phi)$ and every $t > 0$ and for every $p, q \in D$.

$\phi\{N(Kp, Lq, t), N(TRp, MSq, t), N(Lq, MSq, t), N(Kp, TRp, t), N(Kp, MSq, t), N(TRp, Lq, t)\} \geq 0$

(iii) If $TR(D)$ is a closed subset of D.

(iv) Pairs (K, TR) and (L, MS) are weakly compatible.

(v) Each pair of pairs (K, TR) and (L, MS) has a coincidence point in D.

(iv) If $(K, S), (L, R), (MS, R)$ and (TR, S) are commuting pairs than K, L, M, T, R, S have a unique common fixed point in D.

Example 3.8

Let $D = [0, \infty)$ be the fuzzy 2- Banach space.

Define $K, L, M, T, R, S, : D \to D$ by

$$KD = \begin{cases} p \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}; \qquad LD = \begin{cases} \frac{3-p}{2} \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}; \qquad RD = \begin{cases} \frac{4-p}{2} \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}$$

$$SD = \begin{cases} \frac{5-p}{2} \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}; \qquad MD = \begin{cases} \frac{p+1}{2} \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}; \qquad TD = \begin{cases} \frac{p+2}{2} \ if \ p = 1 \\ 1 \ if \ p > 1 \\ 0 \ if \ p < 1 \end{cases}$$

Let $p \in D$.

Consider the sequence $\{p_n\} = \{1 + \frac{1}{n}\}$ and $\{q_n\} = \{1 + \frac{2}{n}\}$.

Here (K, TR) and (L, MS) satisfies the CLR property and are weakly compatible.

All the conditions of the above theorem are satisfied.

1 is the coincidence point in D.

Hence 1 is the unique common fixed point of K, L, M, T, R and S in D respectivly.

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4. Conclusion:

The target of this paper is to emphasize the role of CLR property in the existence of common fixed points in fuzzy 2- Banach space and prove our main results for the pair of weak compatible mapping along with CLR property.

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