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Quasi discreteness analysis of a 2-dimensional Heisenberg ferromagnetic spin system with biquadratic interactions

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Abstract: The dynamical theory of soliton excitations in two-dimensional ferromagnets is studied by introducing a Hamiltonian that includes biquadratic interaction along with uniaxial anisotropy. To obtain a dynamical equation of motion, we use the Dyson-Maleev transformation and the coherent state ansatz. We obtain a Nonlinear Schrödinger equation by applying the multiple-scale and quasi-discreteness methods. For the more general nonintegrable case, we perform a multiple scale perturbation analysis to determine the discreteness effect on the soliton excitations. Finally, we examine modulational instability (MI) under the influence of small perturbations.

1. Introduction

In the past few years, solitons have been extensively studied as nonlinear excitations of magnetic systems [1-5]. There have been several effective theoretical methods developed and applied to a wide range of nonlinear excitations in magnets [6-13]. A one-dimensional spin system can be successfully realized using the semi-classical approach. Combining the Holstein-Primakoff or Dyson-Maleev transformation with coherent state method is a basic semi-classical approach. In magnetic systems, many different nonlinear excitations have been detected using this semi-classical approach, including intrinsic self-localized modes [14], self-induced gap solitons [15] and non-propagation kinks [16].

Research in the nonlinear domain both theoretical [17-19] and experimental [20-22] has recently focused on magnetic soliton dynamics of the Heisenberg spin system with different types of interactions. The biquadratic exchange interaction is one of the most commonly used magnetic interactions in describing magnetic properties and spin excitations among the dominant magnetic interactions, such as bilinear, weak ferromagnetic, and anisotropic interactions [23-26]. The biquadratic exchange interaction plays a major role, and there is considerable interest in the study of ferromagnetic spin chains that combine bilinear and biquadratic exchange interactions [18]. Many researchers have studied soliton in 1D Heisenberg ferromagnetic spin systems with biquadratic interactions [27-30]. It has also been reported that in the semi-classical limit, soliton dynamics can be achieved in a 2D model of a Heisenberg ferromagnetic spin system with bilinear and biquadratic interactions [31]. A recent work by Zhu et al.[23] on iron-based superconductors demonstrated that a biquadratic term describes the magnetism and spin excitations accurately. This has motivated us to study the excitation of soliton in a square lattice model of the biquadratic ferromagnetic spin system and the dynamics are analyzed semi-classically using D-M transformations of bosonic operators and quasi-discreteness combined with multiple scales.

Modulational instability is an extremely important concept in nonlinear wave theory [32-36]. In nonlinear oscillator lattices, Kivshar and Peyrard [37] first suggested that modulational instability could generate localized states. In ferromagnetic chains, modulational instability and nonlinear localized modes have received much attention. It has recently been reported by Kavitha et al.[38] that octupole-dipole and dipole-dipole interactions excite higher-order modulational instability of plane carrier waves. In a recent paper, Latha et al. [39] presented an analysis of modulational instability in the ferromagnetic spin system of a two-dimensional square lattice. However, no studies have been conducted on the stability of our current model under D-M transformation. With these considerations in mind, in this paper, we also investigate the stability of solitons in a 2D biquadratic ferromagnetic spin system.

Hence, in this paper, we construct a model Hamiltonian for the FM system considering biquadratic interactions. Soliton excitations in the system are investigated using Dyson-Maleev transformation, coherent state ansatz, and perturbation technique. Linear stability analysis is also used to analyze the stability aspects of the system.

The paper is structured as follows. In section 2, we derive the dynamical equation of motion for the FM spin chain by considering the model Hamiltonian with biquadratic interactions. In section 3, we use a quasi-disceteness approximation along with multiple scales to develope the Nonlinear schrödinger equation. This equation can be solved using the multiple scale perturbation analysis and the details are given in section 4. In section 5, MI of the FM spin system is discussed, and section 6 provides conclusions.

2. Equations of motion based on the model

For a ferromagnetic spin system with biquadratic and anisotropic interactions we write the Hamiltonian [40] as follows

$$\tilde{H} = -\sum_{n,m} [\tilde{J}'(\vec{S}_{n,m}.\vec{S}_{n+1,m})^2 + \tilde{J}'_1(\vec{S}_{n,m}.\vec{S}_{n,m+1})^2 + \tilde{J}'_2(\vec{S}_{n,m}.\vec{S}_{n+1,m+1})^2 \\ -\tilde{A}(S^z_{n,m})^4]. \tag{1}$$

$\tilde{J}'$, $\tilde{J}'_1$ and $\tilde{J}'_2$ are constant coefficients of biquadratic interactions, whereas $\tilde{A}$ is an anisotropy interaction between crystal fields. As a dimensionless Hamiltonian, we have $\hat{S}_i = \vec{S}_{n,m}/\hbar$ with $\hat{S}^{\pm}_{n,m} = \hat{S}^x_{n,m} \pm i\hat{S}^y_{n,m}$. Spin Hamiltonian (1) now can be expressed as follows:

$$\begin{aligned} H = -\sum_{n,m} [&\tfrac{J'}{4S^4}(\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m}\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m} + \hat{S}^+_{n,m}.\hat{S}^-_{n+1,m}\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m} \\ &+\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m}\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m} + \hat{S}^-_{n,m}.\hat{S}^+_{n+1,m}\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m} + 4\hat{S}^+_{n,m}. \\ &\hat{S}^-_{n+1,m}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m} + 4\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m} + 4\hat{S}^z_{n,m}. \\ &\hat{S}^z_{n+1,m}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m}) + \tfrac{J'_1}{4S^4}(\hat{S}^+_{n,m}.\hat{S}^-_{n,m+1}\hat{S}^+_{n,m}.\hat{S}^-_{n,m+1} + \hat{S}^+_{n,m}. \\ &\hat{S}^-_{n,m+1}\hat{S}^-_{n,m}.\hat{S}^+_{n,m+1} + \hat{S}^-_{n,m}.\hat{S}^+_{n,m+1}\hat{S}^+_{n,m}.\hat{S}^-_{n,m+1} + \hat{S}^-_{n,m}.\hat{S}^+_{n,m+1}\hat{S}^-_{n,m}. \\ &\hat{S}^+_{n,m+1} + 4\hat{S}^+_{n,m}.\hat{S}^-_{n,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n,m+1} + 4\hat{S}^-_{n,m}.\hat{S}^+_{n,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n,m+1} \\ &+4\hat{S}^z_{n,m}.\hat{S}^z_{n,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n,m+1}) + \tfrac{J'_2}{4S^4}(\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m+1}\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m+1} \\ &+\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m+1}\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m+1} + \hat{S}^-_{n,m}.\hat{S}^+_{n+1,m+1}\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m+1} + \\ &\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m+1}\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m+1} + 4\hat{S}^+_{n,m}.\hat{S}^-_{n+1,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m+1} + \\ &4\hat{S}^-_{n,m}.\hat{S}^+_{n+1,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m+1} + 4\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m+1}.\hat{S}^z_{n,m}.\hat{S}^z_{n+1,m+1}) \\ &-\tfrac{A'}{S^4}(\hat{S}^4]. \end{aligned} \tag{2}$$

A Dyson-Maleev representation [41,42] of spin operators can be used to understand spin dynamics of 2D ferromagnetic spin sytems by

$$\hat{S}^+_{n,m} = (2S)^{1/2}[1 - \frac{a^\dagger_{n,m}.a_{n,m}}{2S}]a_{n,m}, \tag{3}$$

$$\hat{S}^-_{n,m} = (2S)^{1/2}a^\dagger_{n,m}, \tag{4}$$

$$\hat{S}^z_{n,m} = S - a^\dagger_{n,m}a_{n,m}. \tag{5}$$

By using bosonic operators, we can satisfy the commutation relations. A Glauber coherent representation [8] of Bose operators has been used to average the eigenstates of the annihilation operator, $\langle u|a^\dagger_{n,m} = \langle u|a^*_{n,m}$ and $a_{n,m}|u\rangle = u_{n,m}|u\rangle$ where $|u\rangle = \Pi_{n,m}|u_{n,m}\rangle$ with $\langle u|u\rangle = 1$, and where $u_{n,m}$ representing the coherent state wave function. For the average $\langle u|a_{n,m|u}\rangle$ of the system in a coherent state, the equation of motion is given as follows:

$$
\begin{aligned}
i\frac{\partial u_{n,m}}{\partial t} &= \frac{J'}{S}[4u_{n,m}-2u_{n-1,m}-2u_{n+1,m}]+\frac{J'}{S^2}A_1+\frac{J'}{S^3}A_2+\frac{J'}{4S^4}A_3 \\
&+\frac{J_1'}{S}[4u_{n,m}-2u_{n,m-1}-2u_{n,m+1}]+\frac{J_1'}{S^2}A_4+\frac{J_1'}{S^3}A_5+\frac{J_1'}{4S^4}A_6 \\
&+\frac{J_2'}{S}[4u_{n,m}-2u_{n-1,m-1}-2u_{n+1,m+1}]+\frac{J_2'}{S^2}A_7+\frac{J_2'}{S^3}A_8+ \\
&\frac{J_2'}{4S^4}A_9-\frac{4A}{S}u_{n,m}+\frac{12A}{S^2}u_{n,m}|u_{n,m}|^2-\frac{12A}{S^3}u_{n,m}^{*}u_{n,m}^{2}|u_{n,m}|^2 \\
&+\frac{4A}{S^4}u_{n,m}^{*2}u_{n,m}^{3}|u_{n,m}|^2. \qquad (6)
\end{aligned}
$$

The coefficients of A_1 - A_9 in Eq.(6) are given in Appendix. Eq. (6) describes the spin dynamics of a (2+1) dimensional ferromagnet with anisotropy. Here, the Planck's constant $\hbar$ is equal to one.

3. Quasi-discreteness approximation

As a consequence of the multiple scale and quasi-discreteness methods [43], we assume the following

$$
\begin{aligned}
u_{n,m}(t) &= \varepsilon u^{(1)}(\xi_n,\zeta_m,\tau,\Phi_n,\Phi_m)+\varepsilon^2 u^{(2)}(\xi_n,\zeta_m,\tau,\Phi_n,\Phi_m) \\
&+\varepsilon^3 u^{(3)}(\xi_n,\zeta_m,\tau,\Phi_n,\Phi_m)+\ldots,=\sum_{\gamma=1}^{\infty}\varepsilon^{\gamma}u_{nm,nm}^{(\gamma)}. \qquad (7)
\end{aligned}
$$

In this case, ε specifies the relative amplitude of the excitation as a small ordering parameter and $\xi_n = \varepsilon(na-\lambda t)$, $\zeta_m=\varepsilon(ma-\eta t)$ and $\tau=\varepsilon^2 t$. There are Multiple scales slow variables, including $\Phi_n = nK_x a - \omega t$ and $\Phi_m = mK_y b - \omega t$. Fast variables $\Phi_n = nK_x a - \omega t$ and $\Phi_m = mK_y b - \omega t$ represent the phase of the carrier wave, K_x, K_y, and ωt indicate its wavenumbers and frequency. Where a, and b are the lattice constants while the parameters λ and η are yet to be revealed. The Taylor approximations for 2 dimensions are modified as follows

$$
\begin{aligned}
u_{n\pm1,m} &= \varepsilon u_{nm,(n\pm1)m}^{(1)} \pm \varepsilon^2 a\frac{\partial u_{nm,(n\pm1)m}^{(1)}}{\partial\xi_n}+\frac{\varepsilon^3 a^2}{2}\frac{\partial^2 u_{nm,(n\pm1)m}^{(1)}}{\partial\xi_n^2} \\
&+\varepsilon^2 u_{nm,(n\pm1)m}^{(2)} \pm \varepsilon^3 a\frac{\partial u_{nm,(n\pm1)m}^{(2)}}{\partial\xi_n}+\varepsilon^3 u_{nm,(n\pm1)m}^{(3)}+\ldots, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
u_{n,m\pm1} &= \varepsilon u_{nm,n(m\pm1)}^{(1)} \pm \varepsilon^2 b\frac{\partial u_{nm,n(m\pm1)}^{(1)}}{\partial\zeta_m}+\frac{\varepsilon^3 b^2}{2}\frac{\partial^2 u_{nm,n(m\pm1)}^{(1)}}{\partial\zeta_m^2}+ \\
&+\varepsilon^2 u_{nm,n(m\pm1)}^{(2)} \pm \varepsilon^3 b\frac{\partial u_{nm,n(m\pm1)}^{(2)}}{\partial\zeta_m}+\varepsilon^3 u_{nm,n(m\pm1)}^{(3)}+\ldots, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
u_{n\pm1,m\pm1} &= \varepsilon u_{nm,(n\pm1)m}^{(1)}+\varepsilon u_{nm,n(m\pm1)}^{(1)} \pm \varepsilon^2 a\frac{\partial u_{nm,(n\pm1)m}^{(1)}}{\partial\xi_n} \pm \varepsilon^2 b \\
&\frac{\partial u_{nm,n(m\pm1)}^{(1)}}{\partial\zeta_m}+\frac{\varepsilon^3 a^2}{2}\frac{\partial^2 u_{nm,(n\pm1)m}^{(1)}}{\partial\xi_n^2}+\frac{\varepsilon^3 b^2}{2}\frac{\partial^2 u_{nm,n(m\pm1)}^{(1)}}{\partial\zeta_m^2} \\
&+\varepsilon^3 ab\frac{\partial^2 u_{nm,(n\pm1)(m\pm1)}^{(1)}}{\partial\xi_n\,\partial\zeta_m}+\varepsilon^2 u_{nm,(n\pm1)m}^{(2)}+\varepsilon^2 u_{nm,n(m\pm1)}^{(2)} \\
&\pm\varepsilon^3 a\frac{\partial u_{nm,(n\pm1)m}^{(2)}}{\partial\xi_n} \pm \varepsilon^3 b\frac{\partial u_{nm,n(m\pm1)}^{(2)}}{\partial\zeta_m}+\varepsilon^3 u_{nm,(n\pm1)m}^{(3)} \\
&+\varepsilon^3 u_{nm,n(m\pm1)}^{(3)}+\ldots. \qquad (10)
\end{aligned}
$$

We achieve the following transformation using the chain method,

$$
\frac{d}{dt} = 2\frac{\partial}{\partial t}-\lambda\varepsilon\frac{\partial}{\partial\xi_n}-\eta\varepsilon\frac{\partial}{\partial\zeta_m}+\varepsilon^2\frac{\partial}{\partial\tau} \qquad (11)
$$

As a result of substituting Eq. (8)-(11) in to Eq. (6), we receive

$$2i\varepsilon\frac{\partial u^{(1)}_{nm,nm}}{\partial t}-i\lambda\varepsilon^2\frac{\partial u^{(1)}_{nm,nm}}{\partial \xi_n}-i\eta\varepsilon^2\frac{\partial u^{(1)}_{nm,nm}}{\partial \zeta_m}+\varepsilon^3\frac{\partial u^{(1)}_{nm,nm}}{\partial \tau}+2i\varepsilon^2\frac{\partial u^{(2)}_{nm,nm}}{\partial t}$$
$$-i\lambda\varepsilon^3\frac{\partial u^{(2)}_{nm,nm}}{\partial \xi_n}-i\eta\varepsilon^3\frac{\partial u^{(2)}_{nm,nm}}{\partial \zeta_m}+2i\varepsilon^3\frac{\partial u^{(3)}_{nm,nm}}{\partial t}=\frac{J'}{S}F_1+\frac{J'}{S^2}F_2+$$
$$\frac{J'_1}{S}F_3+\frac{J'_1}{S^2}F_4+\frac{J'_2}{S}F_5+\frac{J'_2}{S^2}F_6-\frac{4A}{S}(\varepsilon u^{(1)}_{nm,nm}+\varepsilon^2u^{(2)}_{nm,nm}+$$
$$\varepsilon^3u^{(3)}_{nm,nm})+\frac{12A}{S^2}(\varepsilon^3u^{(1)}_{nm,nm}u^{(1)}_{nm,nm}u^{(1)*}_{nm,nm}) \quad (12)$$

Since the coefficients F_1 - F_6 in Eq.(12) are lengthy, they are not presented explicitly in the equation. As a result of comparing the powers of ε in Eq. (12), we obtain the following equations

$$\varepsilon^1: 2i\frac{\partial u^{(1)}_{nm,nm}}{\partial t}-\frac{J'}{S}(4u^{(1)}_{nm,nm}-2u^{(1)}_{nm,(n-1)m}-2u^{(1)}_{nm,(n+1)m})-\frac{J'_1}{S}(4$$
$$u^{(1)}_{nm,nm}-2u^{(1)}_{nm,n(m-1)}-2u^{(1)}_{nm,n(m+1)})-\frac{J'_2}{S}(4u^{(1)}_{nm,nm}-2u^{(1)}_{nm,(n-1)m}$$
$$-2u^{(1)}_{nm,n(m-1)}-2u^{(1)}_{nm,(n+1)m}-u^{(1)}_{nm,n(m+1)})+\frac{4A}{S}u^{(1)}_{nm,nm}=0. \quad (13)$$

$$\varepsilon^2: 2i\frac{\partial u^{(2)}_{nm,nm}}{\partial t}-\frac{J'}{S}(4u^{(2)}_{nm,nm}-2u^{(2)}_{nm,(n-1)m}-2u^{(2)}_{nm,(n+1)m})-\frac{J'_1}{S}(4$$
$$u^{(2)}_{nm,nm}-2u^{(2)}_{nm,n(m-1)}-2u^{(2)}_{nm,n(m+1)})-\frac{J'_2}{S}(4u^{(2}_{nm,nm}-2u^{(2)}_{nm,(n-1)m}$$
$$-2u^{(2)}_{nm,n(m-1)}-2u^{(2)}_{nm,(n+1)m}-2u^{(2)}_{nm,n(m+1)})+\frac{4A}{S}u^{(2)}_{nm,nm}=i\lambda$$
$$\frac{\partial u^{(1)}_{nm,(nm}}{\partial \xi_n}+i\eta\frac{\partial u^{(1)}_{nm,nm}}{\partial \zeta_m}+\frac{J'}{S}(2a\frac{\partial u^{(1)}_{nm,(n-1)m}}{\partial \xi_n}-2a\frac{\partial u^{(1)}_{nm,(n+1)m}}{\partial \xi_n})+$$
$$\frac{J'_1}{S}(2b\frac{\partial u^{(1)}_{nm,n(m-1)}}{\partial \zeta_m}-2b\frac{\partial u^{(1)}_{nm,n(m+1)}}{\partial \zeta_m})+\frac{J'_2}{S}(2a\frac{\partial u^{(1)}_{nm,(n-1)m}}{\partial \xi_n}$$
$$+2b\frac{\partial u^{(1)}_{nm,n(m-1)}}{\partial \zeta_m}-2a\frac{\partial u^{(1)}_{nm,(n+1)m}}{\partial \xi_n}-2b\frac{\partial u^{(1)}_{nm,n(m+1)}}{\partial \zeta_m}). \quad (14)$$

$$\varepsilon^3: 2i\frac{\partial u^{(3)}_{nm,nm}}{\partial t}-\frac{J'}{S}(4u^{(3)}_{nm,nm}-2u^{(2)}_{nm,(n-1)m}-2u^{(3)}_{nm,(n+1)m})-\frac{J'_1}{S}(4$$
$$u^{(3)}_{nm,nm}-2u^{(3)}_{nm,n(m-1)}-2u^{(3)}_{nm,n(m+1)})-\frac{J'_2}{S}(4u^{(3}_{nm,nm}-2u^{(3)}_{nm,(n-1)m}$$
$$-2u^{(3)}_{nm,n(m-1)}-2u^{(3)}_{nm,(n+1)m}-2u^{(3)}_{nm,n(m+1)})+\frac{4A}{S}u^{(3)}_{nm,nm}=-i$$
$$\frac{\partial u^{(1)}_{nm,nm}}{\partial \tau}+i\lambda\frac{\partial u^{(2)}_{nm,nm}}{\partial \xi_n}+i\eta\frac{\partial u^{(2)}_{nm,nm}}{\partial \zeta_m}+\frac{J'}{S}F_7+\frac{J'}{S^2}F_8+\frac{J'_1}{S}F_9+$$
$$\frac{J'_1}{S^2}F_{10}+\frac{J'_2}{S}F_{11}+\frac{J'_2}{S^2}F_{12}-\frac{12A}{S^2}(u^{(1)}_{nm,nm)}u^{(1)}_{nm,nm}u^{(1)*}_{nm,nm}). \quad (15)$$

The coefficients F_7 - F_{12} in Eq. (15) are a much lengthy range, so we do not explicitly provide them. Eq. (13) yields the following solution

$$u^{(1)}_{nm,nm}=B_1e^{i(\phi_n+\phi_m)}+B_1^*e^{-i(\phi_n+\phi_m)}. \quad (16)$$

It is found that the dispersion relation and the group velocity are

$$\omega=\frac{1}{S}[J'(1-cosk_xa+J'_1(1-cosk_yb)+J'_2(1-cosk_xa-cosk_yb)-A], \quad (17)$$

$$V_g=\frac{\partial\omega}{\partial k}=\frac{1}{S}[J'(asink_xa+J'_1(bsink_yb)+J'_2(asink_xa+bsink_yb)-A]. \quad (18)$$

When we substitute Eq. (16) with Eq. (14), we get

$$2i\frac{\partial u^{(2)}_{nm,nm}}{\partial t}-\frac{J'}{S}[4u^{(2)}_{nm,nm}-2u^{(2)}_{nm,(n-1)m}-2u^{(2)}_{nm,(n+1)m}]-\frac{J'_1}{S}[4u^{(2)}_{nm,nm}$$

$$-2u^{(2)}_{nm,n(m-1)} - 2u^{(2)}_{nm,n(m+1)}] - \frac{J'_2}{S}[4u^{(2}_{nm,nm} - 2u^{(2)}_{nm,(n-1)m} - 2u^{(2)}_{nm,n(m-1)}$$
$$-2u^{(2)}_{nm,(n+1)m} - 2u^{(2)}_{nm,n(m+1)}] + \frac{4A}{S}u^{(2)}_{nm,nm} = i\lambda\frac{\partial A}{\partial \xi_n}e^{i(\phi_n+\phi_m)} + i\eta\frac{\partial A}{\partial \zeta_m}$$
$$e^{i(\phi_n+\phi_m)} - i\frac{4J'}{S}asink_xa\frac{\partial A}{\partial \xi_n}e^{i(\phi_n+\phi_m)} - i\frac{4J'_1}{S}bsink_yb\frac{\partial A}{\partial \zeta_m}e^{i(\phi_n+\phi_m)}$$
$$-i\frac{4J'_2}{S}asink_xa\frac{\partial A}{\partial \xi_n}e^{i(\phi_n+\phi_m)} - i\frac{4J'_2}{S}bsink_yb\frac{\partial A}{\partial \zeta_m}e^{i(\phi_n+\phi_m)}]. \quad (19)$$

Eq. (19) on the right side has secular terms proportional to $e^{i(\phi_n+\phi_m)}$ that must be removed. Thus we get,

$$2i\frac{\partial u^{(2)}_{nm,nm}}{\partial t} - \frac{J'}{S}[4u^{(2)}_{nm,nm} - 2u^{(2)}_{nm,(n-1)m} - 2u^{(2)}_{nm,(n+1)m}] - \frac{J'_1}{S}[4u^{(2)}_{nm,nm}$$
$$-2u^{(2)}_{nm,n(m-1)} - 2u^{(2)}_{nm,n(m+1)}] - \frac{J'_2}{S}[4u^{(2}_{nm,nm} - 2u^{(2)}_{nm,(n-1)m} - 2u^{(2)}_{nm,n(m-1)}$$
$$-2u^{(2)}_{nm,(n+1)m} - 2u^{(2)}_{nm,n(m+1)}] + \frac{4A}{S}u^{(2)}_{nm,nm} = 0. \quad (20)$$

and

$$[\lambda - \frac{4J'}{S}asink_xa - \frac{4J'_2}{S}asink_xa]\frac{\partial A}{\partial \xi_n} +$$
$$[\eta - \frac{4J'_1}{S}bsink_yb - \frac{4J'_2}{S}bsink_yb]\frac{\partial A}{\partial \zeta_m} = 0. \quad (21)$$

We can write the following equations using Eq. (21)

$\lambda = \frac{4J'}{S}asink_xa + \frac{4J'_2}{S}asink_xa$, and

$\eta = \frac{4J'_1}{S}bsink_yb + \frac{4J'_2}{S}bsink_yb.$

changing the solution of Eq. (15) of the form

$$u^{(2)}_{nm,nm} = C_1e^{i(\phi_n+\phi_m)} + C_1^*e^{-i(\phi_n+\phi_m)}. \quad (22)$$

We can set C_1 to zero since it is the lowest-order solution. Substituting $u^{(1)}_{nm,nm}$ into Eq. (15) we get the following

$$2i\frac{\partial u^{(3)}_{nm,nm}}{\partial t} - \frac{J'}{S}[4u^{(3)}_{nm,nm} - 2u^{(2)}_{nm,(n-1)m} - 2u^{(3)}_{nm,(n+1)m}] - \frac{J'_1}{S}[4$$
$$u^{(3)}_{nm,nm} - 2u^{(3)}_{nm,n(m-1)} - 2u^{(3)}_{nm,n(m+1)}] - \frac{J'_2}{S}[4u^{(3}_{nm,nm} - 2u^{(3)}_{nm,(n-1)m}$$
$$-2u^{(3)}_{nm,n(m-1)} - 2u^{(3)}_{nm,(n+1)m} - 2u^{(3)}_{nm,n(m+1)}] + \frac{4A}{S}u^{(3)}_{nm,nm} = [i\frac{\partial B_1}{\partial \tau}$$
$$+(\frac{2J'}{S}a^2cosk_xa + \frac{2J'_2}{S}a^2cosk_xa)\frac{\partial^2 B_1}{\partial \xi_n^2} + (\frac{2J'_1}{S}b^2cosk_yb + \frac{2J'_2}{S}b^2$$
$$cosk_yb)\frac{\partial^2 B_1}{\partial \zeta_m^2} + (\frac{4J'_2}{S}abcos(k_xa + k_yb))\frac{\partial^2 B_1}{\partial \xi_n\zeta_m}]e^{i(\phi_n+\phi_m)} + [\frac{J'}{S^2}$$
$$(16 - 20cosk_xa + 4cos2k_xa) + \frac{J'_1}{S^2}(16 - 20cosk_yb + 4cos2k_yb) +$$
$$\frac{J'_2}{S^2}(32cosk_xa + 32cosk_yb - 4cos2k_xa - 4cos2k_yb - 8cos(k_xa + k_yb)$$
$$-12cos(k_xa - k_yb) + 6cos(2k_xa + k_yb) + 6cos(2k_yb - k_xa) - 34) +$$
$$\frac{12A}{S^2}]|B_1|^2B_1e^{i(\phi_n+\phi_m)}. \quad (23)$$

When we remove the secular terms $e^{i(\phi_n+\phi_m)}$ from Eq. (23) on the right, we get

$$i\frac{\partial B_1}{\partial \tau} + \gamma_1\frac{\partial^2 B_1}{\partial \xi_n^2} + \gamma_2\frac{\partial^2 B_1}{\partial \zeta_m^2} + \gamma_3\frac{\partial^2 B_1}{\partial \xi_n\zeta_m} + \gamma_4|B_1|^2B_1 = 0, \quad (24)$$

where

$\gamma_1 = \frac{2J'}{S}a^2cosk_xa + \frac{2J'_2}{S}a^2cosk_xa,$

$\gamma_2 = \frac{2J'_1}{S} b^2 cosk_y b + \frac{2J'_2}{S} b^2 cosk_y b$,

$\gamma_3 = \frac{2J'_2}{S} abcos(k_x a + k_y b)$ and

$\gamma_4 = [\frac{J'}{s^2}(16 - 20cosk_x a + 4cos2k_x a) + \frac{2J'_1}{s^2}(16 - 20cosk_y b + 4cos2k_y b) + \frac{J'_2}{S^2}(32cosk_x a + 32cosk_y b - 4cos2k_x a - 4cos2k_y b - 8cos(k_x a + k_y b) - 12cos(k_x a - k_y b) + 6cos(2k_x a + k_y b) + 6cos(2k_y b - k_x a) - 34]$.

Using the multiple-scale method and Eqs. (7) and (16) together , we obtain $B_1 = \frac{u}{\varepsilon}$, $\xi_n = \varepsilon(na - \lambda t) = \varepsilon X_n$, $\zeta_m = \varepsilon(mb - \eta t) = \varepsilon Y_m$ and $\tau = \varepsilon^2 t$. Hence, Eq. (24) can be written as follows:

$$i\frac{\partial u}{\partial \tau} + \gamma_1 \frac{\partial^2 u}{\partial X_n^2} + \gamma_2 \frac{\partial^2 u}{\partial Y_m^2} + \gamma_3 \frac{\partial^2 u}{\partial X_n Y_m} + \gamma_4 |u|^2 u = 0. \tag{25}$$

Eq. (25) describes a Nonlinear Schr$\ddot{o}$dinger equation in (2+1) dimensions. The next section discusses multiple scale perturbation analysis, which can be used to solve this equation.

4. Perturbation analysis

Approximation methods are useful for analyzing nonlinear systems with slight perturbations. The number of perturbed nonlinear integrable models have been successfully solved using the multiple-scale perturbation method [44]. After appropriate transformations and rescaling t and u as $t \to \frac{t}{\gamma_1}$ and $u \to (\sqrt{\frac{2\gamma_1}{\gamma_4}})u$, we rewrite Eq. (25) as follows :

$$iu_t + u_{xx} + 2|u|^2 u + \lambda_1[\frac{\gamma_2}{\gamma_1}u_{yy} + \frac{\gamma_3}{\gamma_1}u_{xy}] = 0, \tag{26}$$

With perturbation parameter $\lambda_1 = \frac{1}{S}$, when $\gamma = 0$, Eq. (26) permits the following solution to the nonlinear Schr$\ddot{o}$dinger equation,

$$u = \eta sech\eta(\theta - \theta_0)exp[i\xi(\theta - \theta_0) + i(\sigma - \sigma_0)]. \tag{27}$$

We write η, ξ, θ_0 and σ_0 are the parameters that may depend on a long time scale $T = \lambda_1 t$ and $\frac{\partial\theta}{\partial t} = -2\xi$, $\frac{\partial\theta}{\partial x} = 1$, $\frac{\partial\sigma}{\partial t} = \eta^2 + \xi^2$ and $\frac{\partial\sigma}{\partial x} = 0$. Eq. (26) has the following solution under quasi-stationary conditions

$$u = \hat{u}(\theta, T; \lambda_1)exp[i\xi(\theta - \theta_0) + i(\sigma - \sigma_0)]. \tag{28}$$

when we add Eq. (28) to Eq. (26), we get

$$-\eta^2\hat{u} + \hat{u}_{\theta\theta} + 2|\hat{u}|^2\hat{u} = \lambda_1 F(\hat{u}), \tag{29}$$

where

$$F(\hat{u}) = i[-\hat{u}_T - (\frac{\gamma_2}{\gamma_1} + \frac{\gamma_3}{\gamma_1})2\xi\hat{u}_\theta] + \hat{u}[\xi_T(\theta - \theta_0) - (\xi\theta_{0T} + \sigma_{0T}) + (\frac{\gamma_2}{\gamma_1} + \frac{\gamma_3}{\gamma_1})\xi^2] - (\frac{\gamma_2}{\gamma_1} + \frac{\gamma_3}{\gamma_1})\hat{u}_{\theta\theta}. \tag{30}$$

Expanding $\hat{u}$ in terms of Poincare-type asymptotic expansion and Keeping terms up to $O(\gamma_1)$, we obtain

$$\hat{u}(\theta, T; \lambda_1) = \hat{u}_0(\theta, T) + \lambda_1\hat{u}_1(\theta, \lambda_1), \tag{31}$$

where

$$\hat{u}_0 = \eta sech\eta(\theta - \theta_0). \tag{32}$$

Eq. (32) is substituted into Eq. (31) and at $O(\lambda_1)$, we obtain

$$-\eta^2\hat{u}_1+\hat{u}_{1\theta\theta}+4|\hat{u}_0|^2\hat{u}_1+2\hat{u}_0{}^2\hat{u}_1{}^*=\lambda_1F(\hat{u}_0), \tag{33}$$

where

$$F(\hat{u}_0)=i[-\hat{u}_{0T}-(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})2\xi\hat{u}_{0\theta}]+\hat{u}_0[\xi_T(\theta-\theta_0)-(\xi\theta_{0T}+\sigma_{0T}) +(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})\xi^2]-(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})\hat{u}_{0\theta\theta}. \tag{34}$$

If $\hat{u}_1=\hat{\phi}_1+i\hat{\psi}_1$ with $\hat{\phi}_1$ and $\hat{\psi}_1$ are real, Eqs. (33) and (34) can be written as a set of equations

$$L_1\hat{\phi}_1=-\eta^2\hat{\phi}_1+\hat{\phi}_{1\theta\theta}+6\hat{u}_0{}^2\hat{\phi}_1=Re(\hat{F}_1), \tag{35}$$

$$L_2\hat{\psi}_1=-\eta^2\hat{\psi}_1+\hat{\psi}_{1\theta\theta}+2\hat{u}_0{}^2\hat{\psi}_1=Im(\hat{F}_1). \tag{36}$$

In this case, L_1 and L_2 are self-adjoint operators and

$$Re(\hat{F}_1)=\hat{u}_0[\xi_T(\theta-\theta_0)-(\xi\theta_{0T}+\sigma_{0T})+(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})\xi^2] -(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})\hat{u}_{0\theta\theta}, \tag{37}$$

and

$$Im(\hat{F}_1)=-\hat{u}_{0T}-(\tfrac{\gamma_2}{\gamma_1}+\tfrac{\gamma_3}{\gamma_1})2\xi\hat{u}_{0\theta}. \tag{38}$$

As $\hat{u}_0$ and $\hat{u}_{0\theta}$ are homogeneous solutions of Eqs. (35) and (36) respectively, the secularity conditions are

$$\int_{-\infty}^{\infty}\hat{u}_{0\theta}Re(\hat{F}_1)d\theta=0 \tag{39}$$

and

$$\int_{-\infty}^{\infty}\hat{u}_0Im(\hat{F}_1)d\theta=0. \tag{40}$$

When we use the values of $\hat{u}_0$, $\hat{u}_{0\theta}$, $Re(\hat{F}_1)$ and $Im(\hat{F}_1)$ in Eqs. (39) and (40) and compute the integrals, we obtain ξ_T=0 and η_T=0, which indicates that the velocity and amplitude of the soliton remain unchanged. In order to determine perturbed solutions, we need to solve Eqs. (33) and (34) with ξ_T=0 and η_T=0 and appropriate initial conditions. Following are the specific solutions that are admissible for the homogeneous part of Eq.(35):

$$\hat{\phi}_{11}=sech\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0), \tag{41}$$

$$\hat{\phi}_{12}=\frac{1}{\eta}[\tfrac{3}{2}\eta(\theta-\theta_0)sech\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)+\tfrac{1}{2}tanh\eta(\theta-\theta_0)sinh\eta (\theta-\theta_0)-sechh\eta(\theta-\theta_0)]. \tag{42}$$

The formula can be used to construct a general solution when we know two particular solutions,

$$\hat{\phi}_1=c_1\hat{\phi}_{11}+c_2\hat{\phi}_{12}-\hat{\phi}_{11}\int\hat{\phi}_{12}Re(\hat{F}_1)d\theta+\hat{\phi}_{12}\int\hat{\phi}_{11}Re(\hat{F}_1)d\theta. \tag{43}$$

where c_1 and c_2 are used as integration constants. Utilizing Eqs. (37), (41) and (42) to calculate the integrals, we get $\hat{\phi}_1$ in the form

$$\hat{\phi}_1=c_1sech\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)+c_2[\tfrac{3}{2}(\theta-\theta_0)sech\eta(\theta-\theta_0)tanh \eta(\theta-\theta_0)+\tfrac{1}{2\eta}tanh\eta(\theta-\theta_0)sinh\eta(\theta-\theta_0)-\tfrac{1}{\eta}sech\eta(\theta-\theta_0)] -sech^3\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)[\tfrac{3}{2}(\xi\theta_{0T}+\sigma_{0T})(\theta-\theta_0)-\tfrac{3}{2}a\xi^2 (\theta-\theta_0)]+sech^5\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)[\tfrac{3}{8}a\eta(\theta-\theta_0)+\tfrac{3}{8}a\eta^2 (\theta-\theta_0)]-sech\eta(\theta-\theta_0)tanh^2\eta(\theta-\theta_0)[-\tfrac{3}{4}(\xi\theta_{0T}+\sigma_{0T})+ \tfrac{3}{2\eta}(\xi\theta_{0T}+\sigma_{0T})+\tfrac{3}{4}a\xi^2-\tfrac{3}{2\eta}a\xi^2-\tfrac{1}{2}a\eta]-sech\eta(\theta-\theta_0)tanh^3 \eta(\theta-\theta_0)[\tfrac{3}{4}a\eta]-sech\eta(\theta-\theta_0)tanh^4\eta(\theta-\theta_0)[\tfrac{1}{4}a\eta]-sech\eta (\theta-\theta_0)tanh\eta(\theta-\theta_0)[-\tfrac{1}{2\eta}(\xi\theta_{0T}+\sigma_{0T})\theta+\tfrac{1}{2}a\xi^2\eta\theta+a\eta^2\theta]$$

$$+sech^3\eta(\theta-\theta_0)[\tfrac{1}{2\eta}(\xi\theta_{0T}+\sigma_{0T})-\tfrac{1}{2\eta}a\xi^2]-sech^5\eta(\theta-\theta_0)$$
$$[\tfrac{1}{4}a\eta]-sech^2\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)sinh\eta(\theta-\theta_0)[\tfrac{1}{4\eta}(\xi\theta_{0T}$$
$$+\sigma_{0T})-\tfrac{1}{4\eta}a\xi^2]+sech^4\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)sinh\eta(\theta-\theta_0)$$
$$[\tfrac{1}{8}a\eta]-tanh^5\eta(\theta-\theta_0)sinh\eta(\theta-\theta_0)[\tfrac{1}{8}a\eta]. \tag{44}$$

The secular terms (term proportional to $sinh\eta(\theta-\theta_0)$) can be removed by Selecting the arbitrary constant c_2=0. Furthermore, we can apply the boundary conditions $\hat{\phi}_1|_{\theta=\theta_0=0}$ =0 and $\hat{\phi}_{1\theta}|_{\theta=\theta_0=0}$ =0, to obtain c_1=0 and $\xi\theta_{0T}+\sigma_{0T}$=$a\xi^2$. Therefore, the general solution $\hat{\phi}_1$ resembles the following

$$\hat{\phi}_1=[\tfrac{3}{8}a\eta(\theta-\theta_0)-\tfrac{3}{8}a\eta^2(\theta-\theta_0)]sech^5\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)+[\tfrac{1}{4}a\eta]$$
$$sech\eta(\theta-\theta_0)tanh^2\eta(\theta-\theta_0)+[\tfrac{3}{4}a\eta]sech\eta(\theta-\theta_0)tanh^3\eta(\theta-\theta_0)$$
$$+[\tfrac{1}{4}a\eta]sech\eta(\theta-\theta_0)tanh^4\eta(\theta-\theta_0)-[a\eta^2\theta]sech\eta(\theta-\theta_0)tanh\eta$$
$$(\theta-\theta_0)-[\tfrac{1}{4}a\eta]sech^5\eta(\theta-\theta_0). \tag{45}$$

As a result of solving the homogeneous part of Eq. (36) similarly, the following solutions can be obtained.

$$\hat{\psi}_{11}=sech\eta(\theta-\theta_0), \tag{46}$$

$$\hat{\psi}_{12}=\frac{1}{2\eta}[\eta(\theta-\theta_0)sech\eta(\theta-\theta_0)+sinh\eta(\theta-\theta_0)]. \tag{47}$$

It can be shown that Eq. (36) has the following general solution

$$\hat{\psi}_1=c_3\hat{\psi}_{11}+c_4\hat{\psi}_{12}-\hat{\psi}_{11}\int\hat{\psi}_{12}Im(\hat{F}_1)d\theta+\hat{\psi}_{12}\int\hat{\psi}_{11}Im(\hat{F}_1)d\theta. \tag{48}$$

c_3 and c_4 are the integration constants. Analysing the integrals and applying Eqs. (38), (46) and (47) to Eq. (48) yield

$$\hat{\psi}_1=c_3sech\eta(\theta-\theta_0)+c_4[\tfrac{1}{2}(\theta-\theta_0)sech\eta(\theta-\theta_0)+\tfrac{1}{2\eta}sinh\eta(\theta-\theta_0)]$$
$$+[\tfrac{1}{4}(\theta-\theta_0)_T+\tfrac{1}{2}a\xi]sech\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)+sech^3\eta(\theta-\theta_0)$$
$$[\tfrac{1}{2}\eta(\theta-\theta_0)_T(\theta-\theta_0)+a\xi\eta(\theta-\theta_0)]-sech\eta(\theta-\theta_0)[\tfrac{1}{2}\eta\theta(\theta-\theta_0)_T$$
$$-a\xi\eta\theta]+sech^2\eta(\theta-\theta_0)sinh\eta(\theta-\theta_0)[\tfrac{1}{4}(\theta-\theta_0)_T+\tfrac{1}{2}a\xi]. \tag{49}$$

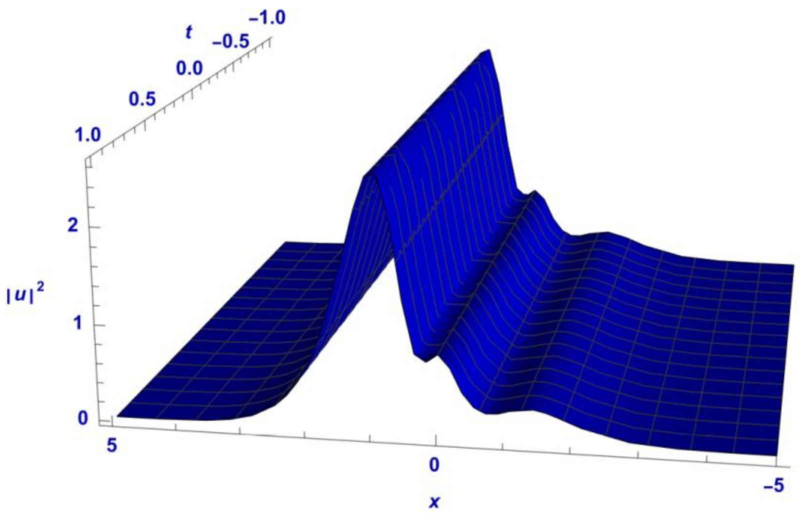

Fig 1: Perturbed soliton with parameters ξ=1.0 and η=1.5.

To eliminate secular terms (term proportional to $sinh\eta(\theta-\theta_0)$), we select an arbitrary constant of c_4=0. We can also take advantage of the boundary conditions $\hat{\psi}_1|_{\theta=\theta_0=0}$ =0 and $\hat{\psi}_{1\theta}|_{\theta=\theta_0=0}$ =0, to obtain c_3=0 and $(\theta-\theta_0)_T$=$\frac{2}{\eta+1}(-a\xi\eta)$. Therefore, the solution $\hat{\psi}_1$ becomes as follows

$$\hat{\psi}_1=[\frac{1}{\eta+1}(a\xi\eta^2\theta+\frac{1}{2}a\xi\eta^2(\theta-\theta_0)]sech\eta(\theta-\theta_0)+[\frac{1}{\eta+1}(-a\xi\eta^2 (\theta-\theta_0)+a\xi\eta(\theta-\theta_0)]sech\eta(\theta-\theta_0)tanh^2\eta(\theta-\theta_0). \quad (50)$$

The perturbed first-order solution $\hat{u}_1$=$\hat{\phi}_1+i\hat{\psi}_1$ can be obtained by applying Eqs. (45) and (50) as

$$\hat{u}_1=[\frac{3}{8}a\eta(\theta-\theta_0)-\frac{3}{8}a\eta^2(\theta-\theta_0)]sech^5\eta(\theta-\theta_0)tanh\eta(\theta-\theta_0)+[\frac{1}{4}a\eta] sech\eta(\theta-\theta_0)tanh^2\eta(\theta-\theta_0)+[\frac{3}{4}a\eta]sech\eta(\theta-\theta_0)tanh^3\eta(\theta-\theta_0) +[\frac{1}{4}a\eta]sech\eta(\theta-\theta_0)tanh^4\eta(\theta-\theta_0)-[a\eta^2\theta]sech\eta(\theta-\theta_0)tanh\eta (\theta-\theta_0)-[\frac{1}{4}a\eta]sech^5\eta(\theta-\theta_0)+i[\frac{1}{\eta+1}(a\xi\eta^2\theta+\frac{1}{2}a\xi\eta^2(\theta-\theta_0)] sech\eta(\theta-\theta_0)+[\frac{1}{\eta+1}(-a\xi\eta^2(\theta-\theta_0)+a\xi\eta(\theta-\theta_0)]sech\eta(\theta-\theta_0) tanh^2\eta(\theta-\theta_0)]. \quad (51)$$

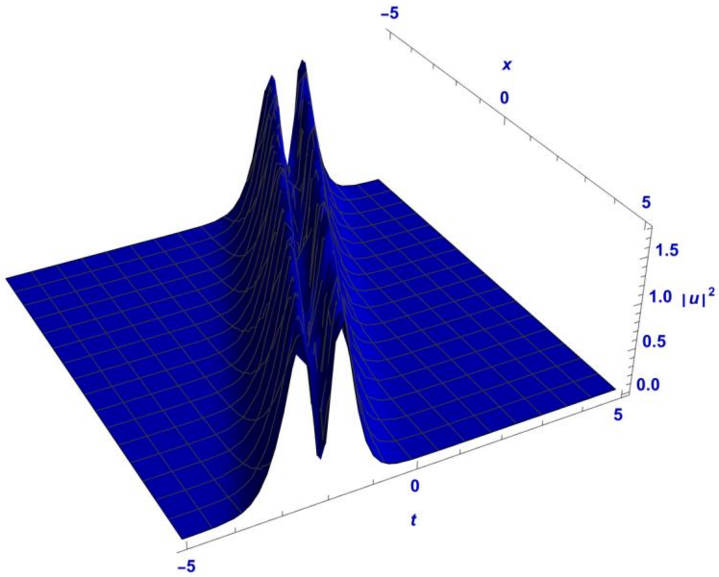

Fig 2: Perturbed soliton with parameters ξ=1.7 and η=2.0

In Fig.1, the perturbed soliton is plotted with constant velocity ξ=1.0 and amplitude η=1.5 and in Fig.2, it is plotted with constant velocity ξ=1.7 and amplitude η=2. According to the figure, variations in η make only very small fluctuations in the localized region without altering the overall solitonic character.

5. Modulational Instability Analysis

By introducing a small phase and amplitude disturbance in a 2D FM spin system, we explore the MI of nonlinear plane waves and soliton formation analytically. In this section, we will discuss the occurrence of MI in the above system. This study examines the time evolution of perturbed nonlinear waves that have the form

$$u_{n,m}(t)=[\phi_0+b_{n,m}(t)]e^{i[\theta_{n,m}(t)+\varphi_{n,m}(t)]}. \quad (52)$$

In the case, where $\theta_{n,m}(t)$=$qn+qm-\omega_0 t$, with ω_0 obeys the nonlinear dispersion relation

$$\omega_0 = [\frac{4J'}{S} + \frac{4J'_1}{S} + \frac{4J'_2}{S} - \frac{4A}{S}] - [\frac{16J'}{S^2} + \frac{16J'_1}{S^2} + \frac{16J'_2}{S^2} - \frac{12A}{S^2}]\phi_0^2 - [\frac{9J'}{S^3} + \frac{9J'_1}{S^3} + \frac{9J'_2}{S^3} - \frac{12A}{S^3}]\phi_0^4 - [\frac{4J'}{S} + \frac{4J'_1}{S} - \frac{20J'}{S^2}\phi_0^2 - \frac{20J'_1}{S^2}\phi_0^2 + \frac{2J'}{S^3}\phi_0^4 + \frac{2J'_1}{S^3}\phi_0^4]cosq - [\frac{4J'_2}{S} + \frac{4J'}{S^2}\phi_0^2 + \frac{4J'_1}{S^2}\phi_0^2 - \frac{2J'}{S^3}\phi_0^4 - \frac{2J'_1}{S^3}\phi_0^4 - \frac{20J'_2}{S^2}\phi_0^2 + \frac{2J'_2}{S^3}\phi_0^4]cos2q - [\frac{4J'_2}{S^2}\phi_0^2 - \frac{2J'_2}{S^3}\phi_0^4]cos4q. \quad (53)$$

Here, the plane waves have a constant amplitude ϕ_0. A system of linearly coupled equations is obtained by substituting Eq.(52) into Eq.(6) and assuming $|b_{n,m}(t)| \ll \phi_0$ and $|\varphi_{n,m}(t)| \ll \theta_{n,m}(t)$, and then taking in to account Eq.(53), is as follows:

$$\frac{\partial b_{nm}}{\partial t} = \frac{J'}{S}\phi_0[4\phi_{nm} - 2\phi_{n-1,m} - 2\phi_{n+1,m}]cosq + \frac{J'_1}{S}\phi_0[4\phi_{nm} - 2\phi_{n,m-1} - 2\phi_{n,m+1}]cosq + \frac{J'_2}{S}\phi_0[4\phi_{nm} - 2\phi_{n-1,m-1} - 2\phi_{n+1,m+1}]cos2q + \frac{J'}{S}[2b_{n-1,m} - 2b_{n+1,m}]sinq + \frac{J'_1}{S}[2b_{n,m-1} - 2b_{n,m+1}]sinq + \frac{J'_2}{S}[2b_{n-1,m-1} - 2b_{n+1,m+1}]sin2q - [\frac{4J'}{S} + \frac{4J'_1}{S} + \frac{4J'_2}{S}]\phi_0\phi_{nm} + \frac{J'}{S^2}\phi_0^2[3b^*_{n-1,m} - 3b^*_{n+1,m}]sinq + \frac{J'_1}{S^2}\phi_0^2[3b^*_{n,m-1} - 3b^*_{n,m+1}]sinq + \frac{J'_2}{S^2}\phi_0^2[3b^*_{n-1,m-1} - 3b^*_{n+1,m+1}]sin2q + \frac{J'}{S^2}\phi_0^3[4\phi^*_{nm} + 4\phi_{nm} - 4\phi_{n-1,m} - 4\phi_{n+1,m}]cos2q + \frac{J'_1}{S^2}\phi_0^3[4\phi^*_{nm} + 4\phi_{nm} - 4\phi_{n,m-1} - 4\phi_{n,m+1}]cos2q + \frac{J'_2}{S^2}\phi_0^3[4\phi^*_{nm} + 4\phi_{nm} - 4\phi_{n-1,m-1} - 4\phi_{n+1,m+1}]cos4q + \frac{J'}{S^2}\phi_0^3[4\phi^*_{nm} - 4\phi_{nm} + 6\phi^*_{n-1,m} - 6\phi_{n-1,m} + 6\phi^*_{n+1,m} - 6\phi_{n+1,m}] + \frac{J'_1}{S^2}\phi_0^3[4\phi^*_{nm} - 4\phi_{nm} + 6\phi^*_{n,m-1} - 6\phi_{n,m-1} + 6\phi^*_{n,m+1} - 6\phi_{n,m+1}] + \frac{J'_2}{S^2}\phi_0^3[4\phi^*_{nm} - 4\phi_{nm} + 6\phi^*_{n-1,m-1} - 6\phi_{n-1,m-1} + 6\phi^*_{n+1,m+1} - 6\phi_{n+1,m+1}] - \frac{J'}{S^3}\phi_0^4[\phi_{n+1,m} + \phi_{n-1,m}]cosq - \frac{J'_1}{S^3}\phi_0^4[\phi_{n,m+1} + \phi_{n,m-1}]cosq - \frac{J'_2}{S^3}\phi_0^4[\phi_{n+1,m+1} + \phi_{n-1,m-1}] - \frac{J'}{S^3}\phi_0^5[2\phi^*_{n+1,m} + 2\phi^*_{n-1,m}]cos2q - \frac{J'_1}{S^3}\phi_0^5[2\phi^*_{n,m+1} + 2\phi^*_{n,m-1}]cos2q - \frac{J'_2}{S^3}\phi_0^5[2\phi^*_{n+1,m+1} + 2\phi^*_{n-1,m-1}]cos4q + \frac{2J'}{S^3}\phi_0^5 b^*_{n,m}sin2q + \frac{2J'_1}{S^3}\phi_0^5 b^*_{n,m}sin2q + \frac{2J'_2}{S^3}\phi_0^5 b^*_{n,m}sin4q. \quad (54)$$

$$-\phi_0\frac{\partial \phi_{nm}}{\partial t} = \frac{J'}{S}[4b_{n,m} - 2b_{n-1,m} - 2b_{n+1,m}]cosq + \frac{J'_1}{S}[4b_{n,m} - 2b_{n,m-1} - 2b_{n,m+1}]cosq + \frac{J'_2}{S}[4b_{n,m} - 2b_{n-1,m-1} - 2b_{n+1,m+1}]cos2q + \frac{J'}{S}\phi_0[2\phi_{n+1,m} - 2\phi_{n-1,m}]sinq + \frac{J'_1}{S}\phi_0[2\phi_{n,m+1} - 2\phi_{n,m-1}]sinq + \frac{J'_2}{S}\phi_0[2\phi_{n+1,m+1} - 2\phi_{n-1,m-1}]sin2q + \frac{J'}{S^2}\phi_0^2[4b_{n-1,m} + 4b_{n+1,m} + 3b^*_{n-1,m} + 3b_{n-1,m} + 3b^*_{n+1,m} + 3b_{n+1,m}]cosq + \frac{J'_1}{S^2}\phi_0^2[4b_{n,m-1} + 4b_{n,m+1} + 3b^*_{n,m-1} + 3b_{n,m-1} + 3b^*_{n,m+1} + 3b_{n,m+1}]cosq + \frac{J'_2}{S^2}\phi_0^2[4b_{n-1,m-1} + 4b_{n+1,m+1} + 3b^*_{n-1,m-1} + 3b_{n-1,m-1} + 3b^*_{n+1,m+1} + 3b_{n+1,m+1}]cos2q + \frac{J'}{S^2}\phi_0^3[10\phi_{n-1,m} - 10\phi_{n+1,m}]sinq + \frac{J'_1}{S^2}\phi_0^3[10\phi_{n,m-1} - 10\phi_{n,m+1}]sinq + \frac{J'_2}{S^2}\phi_0^3[10\phi_{n-1,m-1} - 10\phi_{n+1,m+1}]sin2q + \frac{J'}{S^2}\phi_0^2[4b_{n,m} - 4b^*_{n,m} - 4b_{n-1,m} - 4b_{n+1,m}]cos2q + \frac{J'_1}{S^2}\phi_0^2[4b_{n,m} - 4b^*_{n,m} - 4b_{n,m-1} - 4b_{n,m+1}]cos2q + \frac{J'_2}{S^2}\phi_0^2[4b_{n,m} - 4b^*_{n,m} - 4b_{n-1,m-1} - 4$$

$$b_{n+1,m+1}]cos4q + \frac{J'}{S^2}\phi_0^3[4\phi_{n+1,m} - 4\phi_{n-1,m}]sin2q + \frac{J'_1}{S^2}\phi_0^3 [4\phi_{n,m+1} - 4\phi_{n,m-1}]sin2q + \frac{J'_2}{S^2}\phi_0^3[4\phi_{n+1,m+1} - 4\phi_{n-1,m-1}] sin4q - \frac{J'}{S^2}\phi_0^2[4b_{n,m} + 4b_{n,m}^* - 6b_{n-1,m}^* + 6b_{n-1,m} + 6b_{n+1,m}^* +6b_{n+1,m}] - \frac{J'_1}{S^2}\phi_0^2[4b_{n,m} + 4b_{n,m}^* + 6b_{n,m-1}^* + 6b_{n,m-1} + 6 b_{n,m+1}^* + 6b_{n,m+1}] - \frac{J'_2}{S^2}\phi_0^2[4b_{n,m} + 4b_{n,m}^* + 6b_{n-1,m-1}^* + 6 b_{n-1,m-1} + 6b_{n+1,m+1}^* + 6b_{n+1,m+1}] + \frac{12A}{S^2}[b_{n,m}^* + b_{n,m}] + \frac{J'}{S^3}\phi_0^4[\phi_{n+1,m} - \phi_{n-1,m}]sinq + \frac{J'_1}{S^3}\phi_0^4[\phi_{n,m+1} - \phi_{n,m-1}]sinq + \frac{J'_2}{S^3}\phi_0^4[\phi_{n+1,m+1} - \phi_{n-1,m-1}]sin2q + \frac{J'}{S^3}\phi_0^4[4b_{n,m} + 2b_{n+1,m}^* +2b_{n-1,m}^*]cos2q + \frac{J'_1}{S^3}\phi_0^4[4b_{n,m} + 2b_{n,m+1}^* + 2b_{n,m-1}^*]cos2q + \frac{J'_2}{S^3}\phi_0^4[4b_{n,m} + 2b_{n+1,m+1}^* + 2b_{n-1,m-1}^*]cos4q + \frac{J'}{S^3}\phi_0^5[2\phi_{n-1,m}^* -2\phi_{n+1,m}^*]sin2q + \frac{J'_1}{S^3}\phi_0^5[2\phi_{n,m-1}^* - 2\phi_{n,m+1}^*]sin2q + \frac{J'_2}{S^3}\phi_0^5 [2\phi_{n-1,m-1}^* - 2\phi_{n+1,m+1}^*]sin4q + \frac{12A}{S^3}[2b_{n,m} + 2b_{n,m}^*]. \quad (55)$$

A linear system of this type has the following general solution

$$\begin{pmatrix}\varphi_{nm}\\ b_{nm}\end{pmatrix} = \begin{pmatrix}\varphi_1 & \varphi_2\\ b_1 & b_2\end{pmatrix}\begin{pmatrix}e^{i(Qn+Qm+\Omega t)}\\ e^{-i(Qn+Qm+\Omega^* t)}\end{pmatrix}. \quad (56)$$

where φ_1, φ_2, b_1 and b_2 are the arbitrary real constants, while Q refers to noise wave number and Q^* is its complex conjugate. When We combine solutions Eqs. (54) and (55) we obtain four complex equations. This system produces two sets of real and imaginary parts. As a result of canceling the imaginary part, we have the following system

$$\begin{pmatrix}c_{21}+\sigma_i & c_{22} & c_{23} & c_{24}\\ c_{11} & c_{12}+\sigma_i & c_{13} & c_{14}\\ c_{41} & c_{42} & c_{43}+\sigma_i & c_{44}\\ c_{31} & c_{32} & c_{33} & c_{34}+\sigma_i\end{pmatrix}\begin{pmatrix}\varphi_1\\ b_1\\ \varphi_2\\ b_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \quad (57)$$

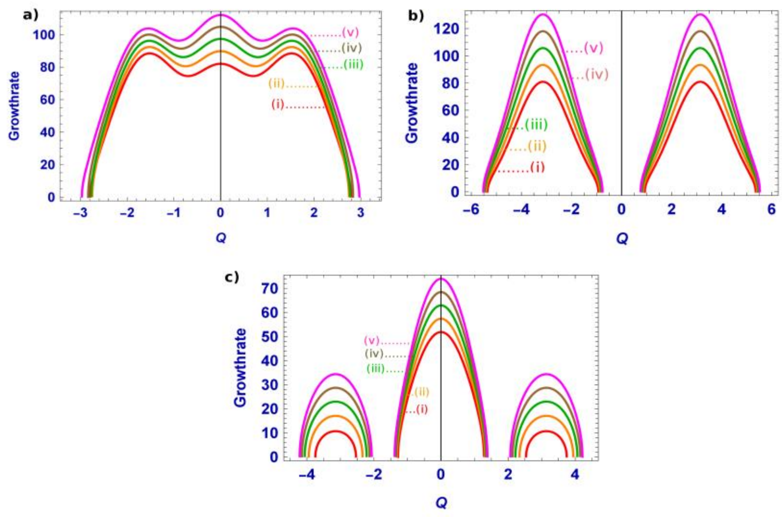

Fig 3: Soliton density profile for various J' values (i) $J' = 6$, (ii) $J' = 8$, (iii) $J' = 10$, (iv) $J' = 12$, (v) $J' = 14$.

A non-trivial solution is available if the determinant vanishes. In this case, the condition is as follows

$$\sigma_i^4 + l_0\sigma_i^2 + l_1 = 0. \tag{58}$$

If $\sigma_i = \phi_0\Omega_{im}$, it determines Ω_{im}. The matrix components are listed in the Appendix . To express analytically the imaginary part of the frequency of the noise modulating the nonlinear wave, the following implicit forms can be used by factorizing Eq.(58)

$$\Omega_{im_1} = \pm\sqrt{\alpha_1(q,Q)} and \Omega_{im_2} = \pm\sqrt{\alpha_2(q,Q)} \tag{59}$$

where $\Omega_{im1}^2.\Omega_{im2}^2.\phi_0^4 = l_1$ and the explicit forms of $\alpha_1(q,Q)$ and $\alpha_2(q,Q)$ can be found in Appendix A. Using the above equations, the real frequency part can be determined by using a diagonal system. When all diagonal terms are equal, this system reduces to a single equation. Therefore, the corresponding real component of the modulated wave frequency Ω_r can be expressed in the following form

$$\Omega_r = (\frac{4J'}{S} + \frac{4J'_1}{S}) sinqsinQ + \frac{J'_2}{S} sin2qsin2Q + (\frac{20J'}{S^2} + \frac{20J'_1}{S^2})\phi_0^2 sinq \\ sinQ + \frac{20J'_2}{S^2}\phi_0^2 sin2qsin2Q - (\frac{8J'}{S^2} + \frac{8J'_1}{S^2})\phi_0^2 sin2qsinQ - \frac{J'_2}{S^2}\phi_0^2 \\ sin4qsin2Q - (\frac{2J'}{S^3} + \frac{2J'_1}{S^3})\phi_0^3 sinqsinQ - \frac{2J'_2}{S^3}\phi_0^3 sin2qsin2Q. \tag{60}$$

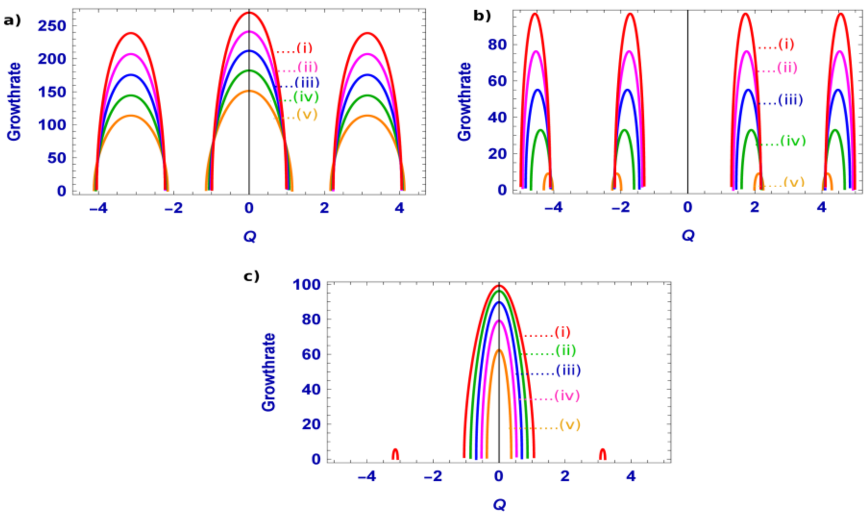

Fig 4: Modulational instability regions for $(a)\ q = 0\ (b)\ q = \pi\ (c)\ q = \frac{\pi}{2}$ with $J'_1 = 6$, (ii) $J'_1 = 8$, (iii) $J'_1 = 10$, (iv) $J'_1 = 12$, (v) $J'_1 = 14$.

Using the imaginary component of Ω presented in Eq.(59), we can determine the stability of extended nonlinear spin waves. Ordinarily, modulational instability (MI) occurs if one of the four imaginary parts of the noise's frequency is non-zero, i.e. $\Omega_{im} = Im(\Omega) \neq 0$. If the imaginary part is not present, MI does not occur. The MI growth rate is given by $G = Im(\Omega)$. Eq.(59) provides the conditions for the MI gain over long wavelengths($q = 0$), short wavelengths($q = \pi$) and the Brillouin boundary zone($q = \pi/2$). The stability diagram and corresponding growth rate are plotted in Figures 3(a) and 3(b) and 3(c) for different values of J'. It seems that the instabilities are larger and the growth rates have increased maximally. It follows that higher biquadratic interactions would lead to stronger modulational instability in materials.

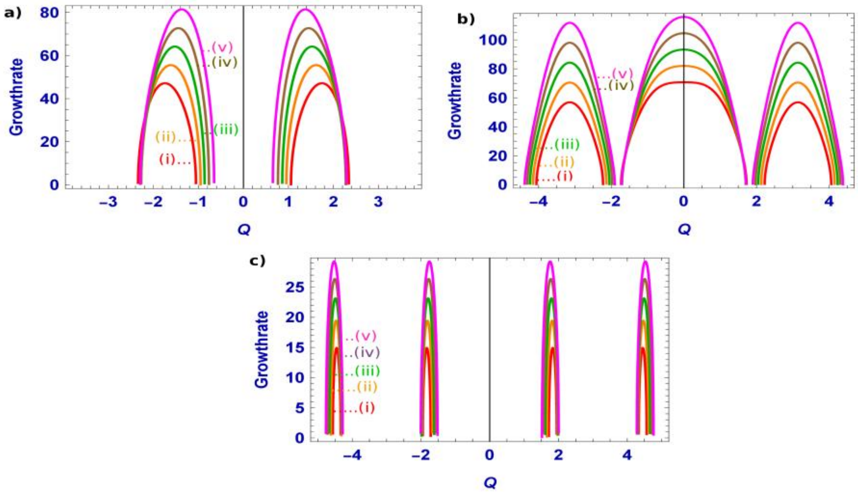

Fig 5: Modulational instability regions for $(a)\ q = 0\ (b)\ q = \pi\ (c)\ q = \frac{\pi}{2}$ with $J'_2 = 8$, (ii) $J'_2 = 10$, (iii) $J'_2 = 12$, (iv) $J'_2 = 14$, (v) $J'_2 = 16$.

Fig.(4) illustrates the growth rates of instability for $q = 0$, $q = \pi$ and $q = \pi/2$ plane waves with varying values of J'_1. We choose $J'_1 = 6$, $J'_1 = 8$, $J'_1 = 10$, $J'_1 = 12$ and $J'_1 = 14$ as biquadratic parameters. According to these figures, if the biquadratic exchange interaction strength is smaller, a given ferromagnetic material can sustain very long-lived excitations generated by MI. Even a reduced stability zone for excited carrier waves can still be achieved if the strength of the biquadratic interaction is higher. With a further increase in the biquadratic parameter, they would ultimately be destroyed. Following that, we analyze the different values of the parameter J'_2 as shown in Fig.(5). It is evident in this figure that as the biquadratic exchange parameter increases, the regions of instability expand and new regions appear. A stability diagram with varying values of the anisotropy parameter is plotted in Figures 6(a)-(c) to illustrate how anisotropy affects the stability of all excitations that may occur within the framework of this study. By looking at the figure, it is obvious that, a larger instability zone results from a larger anisotropy parameter. Also, the instability zone may be reduced by reducing the anisotropy parameter.

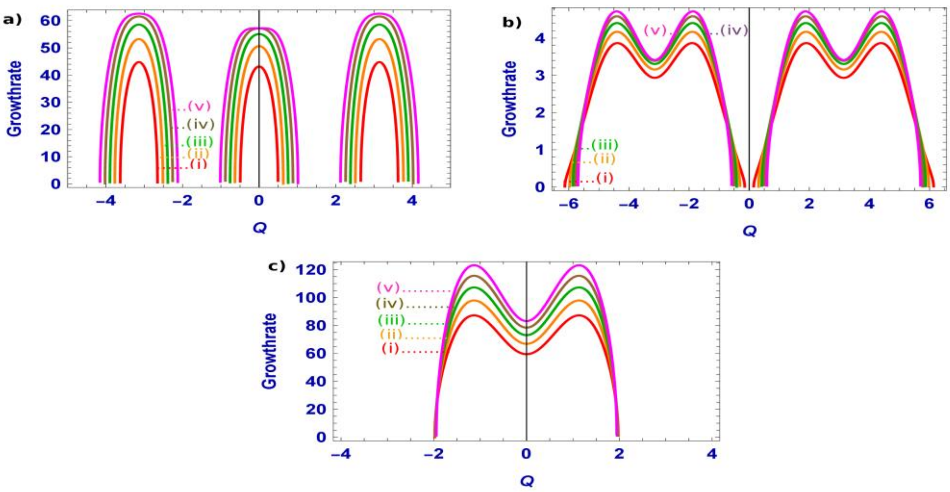

Fig 6: Modulational instability regions for $(a)\ q = 0\ (b)\ q = \pi\ (c)\ q = \frac{\pi}{2}$ with $A = 1$, (ii) $A = 3$, (iii) $A = 5$, (iv) $A = 7$, (v) $A = 9$.

6. Conclusion

Solitary excitations in a two-dimensional FM system with biquadratic and anisotropic interactions are investigated in this paper. To find the spin dynamics, we employ the Dyson-Maleev transformation as well as the coherent state ansatz. By using quasi-discreteness approximation combined with the multiple scale method, we can simplify this equation to the NLS equation. The perturber soliton was constructed using the results of a multiple scale perturbation analysis. As a result of a perturbative analysis, discreteness does not drastically change the nature of the soliton except introducing some fluctuation in the localized region. It is found that soliton moves uniformly under disturbances.

Moreover, MI analysis is carried out using both analytical and graphical methods. By analyzing MI, we have demonstrated that the stability domain can grow or shrink depending on the biquadratic exchange energy and anisotropy energy. Our analytical simulations have shown, therefore, that a nonlinear ferromagnetic chain can become long-lived if the biquadratic parameter is chosen appropriately.

Appendix

The coefficients A_1 - A_9 in Eq.(6) are given by

$$A_1 = [4u_{n-1,m}|u_{n,m}|^2 + 4u_{n+1,m}|u_{n,m}|^2 + 3|u_{n-1,m}|^2u_{n-1,m} + 3|u_{n+1,m}|^2u_{n+1,m} + 3u^*_{n+1,m}u^2_{n,m} + 3u^*_{n-1,m}u^2_{n,m} - 2u^*_{n,m}u^2_{n-1,m} - 2u^*_{n,m}u^2_{n+1,m} - 4u_{n,m}|u_{n,m}|^2 - 6u_{n,m}|u_{n-1,m}|^2 - 6u_{n,m}|u_{n+1,m}|^2],$$

$$A_2 = [u^3_{n,m}u^{*2}_{n+1,m} + u^3_{n,m}u^{*2}_{n-1,m} + 2u^*_{n,m}u^2_{n-1,m}|u_{n-1,m}|^2 + 2u^*_{n,m}u^2_{n+1,m}|u_{n+1,m}|^2 - |u_{n-1,m}|^2|u_{n-1,m}|^2u_{n-1,m} - |u_{n+1,m}|^2|u_{n+1,m}|^2u_{n+1,m} - 2u^*_{n+1,m}u^2_{n,m}|u_{n,m}|^2 - 2u^*_{n-1,m}u^2_{n,m}|u_{n,m}|^2 + 2u_{n,m}|u_{n-1,m}|^2|u_{n-1,m}|^2 + 4u_{n,m}|u_{n,m}|^2|u_{n+1,m}|^2 + 3u_{n,m}|u_{n+1,m}|^2|u_{n+1,m}|^2 + 6u_{n,m}|u_{n,m}|^2|u_{n-1,m}|^2 - 3u^2_{n,m}u^*_{n+1,m}|u_{n+1,m}|^2 - 6u^*_{n,m}u_{n-1,m}u_{n+1,m}|u_{n-1,m}|^2],$$

$$A_3 = [4u^4_{n,m}u^*_{n+1,m}|u_{n+1,m}|^2 + 8|u_{n,m}|^2u^3_{n-1,m}|u_{n-1,m}|^2 + 8u^2_{n,m}|u_{n,m}|^2u^*_{n-1,m}|u_{n-1,m}|^2 + 8|u_{n,m}|^2u^*_{n+1,m}u^2_{n+1,m}|u_{n+1,m}|^2 + 4u_{n,m}|u_{n-1,m}|^2|u_{n-1,m}|^2 + 8u_{n,m}|u_{n,m}|^2|u_{n+1,m}|^2 - u^5_{n,m}u^{*2}_{n+1,m} - 2u^*_{n,m}u^4_{n-1,m}|u_{n-1,m}|^2 - 2u^3_{n,m}u^{*2}_{n-1,m}|u_{n,m}|^2 - 2u^*_{n,m}u^*_{n+1,m}u^3_{n+1,m}|u_{n+1,m}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n-1,m}|^2|u_{n-1,m}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n+1,m}|^2|u_{n+1,m}|^2]],$$

$$A_4 = [4u_{n,m-1}|u_{n,m}|^2 + 4u_{n,m+1}|u_{n,m}|^2 + 3|u_{n,m-1}|^2u_{n,m-1} + 3|u_{n,m+1}|^2u_{n,m+1} + 3u^*_{n,m+1}u^2_{n,m} + 3u^*_{n,m-1}u^2_{n,m} - 2u^*_{n,m}u^2_{n,m-1} - 2u^*_{n,m}u^2_{n,m+1} - 4u_{n,m}|u_{n,m}|^2 - 6u_{n,m}|u_{n,m-1}|^2 - 6u_{n,m}|u_{n,m+1}|^2],$$

$$A_5 = [u^3_{n,m}u^{*2}_{n,m+1} + u^3_{n,m}u^{*2}_{n,m-1} + 2u^*_{n,m}u^2_{n,m-1}|u_{n,m-1}|^2 + 2u^*_{n,m}u^2_{n,m+1}|u_{n,m+1}|^2 - |u_{n,m-1}|^2|u_{n,m-1}|^2u_{n,m-1} - |u_{n,m+1}|^2|u_{n,m+1}|^2u_{n,m+1} - 2u^*_{n,m+1}u^2_{n,m}|u_{n,m}|^2 - 2u^*_{n,m-1}u^2_{n,m}|u_{n,m}|^2 + 2u_{n,m}|u_{n,m-1}|^2|u_{n,m-1}|^2 + 4u_{n,m}|u_{n,m}|^2|u_{n,m+1}|^2 + 3u_{n,m}|u_{n,m+1}|^2|u_{n,m+1}|^2 + 6u_{n,m}|u_{n,m}|^2|u_{n,m-1}|^2 - 3u^2_{n,m}u^*_{n,m+1}|u_{n,m+1}|^2 - 6u^*_{n,m}u_{n,m-1}u_{n,m+1}|u_{n,m-1}|^2],$$

$$A_6 = [4u^4_{n,m}u^*_{n,m+1}|u_{n,m+1}|^2 + 8|u_{n,m}|^2u^3_{n,m-1}|u_{n,m-1}|^2 + 8u^2_{n,m}|u_{n,m}|^2u^*_{n,m-1}|u_{n,m-1}|^2 + 8|u_{n,m}|^2u^*_{n,m+1}u^2_{n,m+1}|u_{n,m+1}|^2 + 4u_{n,m}|u_{n,m-1}|^2|u_{n,m-1}|^2 + 8u_{n,m}|u_{n,m}|^2|u_{n,m+1}|^2 - u^5_{n,m}u^{*2}_{n,m+1} - 2u^*_{n,m}u^4_{n,m-1}|u_{n,m-1}|^2 - 2u^3_{n,m}u^{*2}_{n,m-1}|u_{n,m}|^2 - 2u^*_{n,m}u^*_{n,m+1}u^3_{n,m+1}|u_{n,m+1}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n,m-1}|^2|u_{n,m-1}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n,m+1}|^2|u_{n,m+1}|^2],$$

$$A_7 = [4u_{n-1,m-1}|u_{n,m}|^2 + 4u_{n+1,m+1}|u_{n,m}|^2 + 3|u_{n-1,m-1}|^2u_{n-1,m-1} + 3|u_{n+1,m+1}|^2u_{n+1,m+1} + 3u^*_{n+1,m+1}u^2_{n,m} + 3u^*_{n-1,m-1}u^2_{n,m} - 2u^*_{n,m}u^2_{n-1,m-1} - 2u^*_{n,m}u^2_{n+1,m+1} - 4u_{n,m}|u_{n,m}|^2 - 6u_{n,m}|u_{n-1,m-1}|^2 - 6u_{n,m}|u_{n+1,m+1}|^2],$$

$$A_8 = [u^3_{n,m}u^{*2}_{n+1,m+1} + u^3_{n,m}u^{*2}_{n-1,m-1} + 2u^*_{n,m}u^2_{n-1,m-1}|u_{n-1,m-1}|^2 + 2u^*_{n,m}u^2_{n+1,m+1}|u_{n+1,m+1}|^2 - |u_{n-1,m-1}|^2|u_{n-1,m-1}|^2u_{n-1,m-1} - |u_{n+1,m+1}|^2|u_{n+1,m+1}|^2u_{n+1,m+1} - 2u^*_{n+1,m+1}u^2_{n,m}|u_{n,m}|^2 -$$

$2u^*_{n-1,m-1}u^2_{n,m}|u_{n,m}|^2 + 2u_{n,m}|u_{n-1,m-1}|^2|u_{n-1,m-1}|^2 + 4u_{n,m}|u_{n,m}|^2|u_{n+1,m+1}|^2 + 3u_{n,m}|u_{n+1,m+1}|^2|u_{n+1,m+1}|^2 + 6u_{n,m}|u_{n,m}|^2|u_{n-1,m-1}|^2 - 3u^2_{n,m}u^*_{n+1,m+1}|u_{n+1,m+1}|^2 - 6u^*_{n,m}u_{n-1,m-1}u_{n+1,m+1}|u_{n-1,m-1}|^2],$

$A_9 = [4u^4_{n,m}u^*_{n+1,m+1}|u_{n+1,m+1}|^2 + 8|u_{n,m}|^2u^3_{n-1,m-1}|u_{n-1,m-1}|^2 + 8u^2_{n,m}|u_{n,m}|^2u^*_{n-1,m-1}|u_{n-1,m-1}|^2 + 8|u_{n,m}|^2u^*_{n+1,m+1}u^2_{n+1,m+1}|u_{n+1,m+1}|^2 + 4u_{n,m}|u_{n-1,m-1}|^2|u_{n-1,m-1}|^2 + 8u_{n,m}|u_{n,m}|^2|u_{n+1,m+1}|^2 - u^5_{n,m}u^{*2}_{n+1,m+1} - 2u^*_{n,m}u^4_{n-1,m-1}|u_{n-1,m-1}|^2 - 2u^3_{n,m}u^{*2}_{n-1,m-1}|u_{n,m}|^2 - 2u^*_{n,m}u^*_{n+1,m+1}u^3_{n+1,m+1}|u_{n+1,m+1}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n-1,m-1}|^2|u_{n-1,m-1}|^2 - 12u_{n,m}|u_{n,m}|^2|u_{n+1,m+1}|^2|u_{n+1,m+1}|^2].$

The different components of the matrix in Eq.(57) are

$a_1 = (\frac{4J'}{S} + \frac{4J'_1}{S})\phi_0(1 - cosQ)cosq + \frac{4J'_2}{S}\phi_0 cos2q(1 - cos2Q) - (\frac{4J'}{S} + \frac{4J'_1}{S} + \frac{4J'_2}{S})\phi_0 + (\frac{4J'}{S^2} + \frac{4J'_1}{S^2})\phi_0^3(1 - 2cosQ)cos2q + \frac{4J'_2}{S^2}\phi_0^3(1 - 2cos2Q)cosq - (\frac{4J'}{S^2} + \frac{4J'_1}{S^2} + \frac{4J'_2}{S^2})\phi_0^3 - (\frac{12J'}{S^2} + \frac{12J'_1}{S^2})\phi_0^3 cosq - \frac{12J'_2}{S^2}\phi_0^3 cos2Q + \frac{12A}{S^2}\phi_0^3 - (\frac{2J'}{S^3} + \frac{2J'_1}{S^3})\phi_0^4 cosqcosQ - \frac{2J'_2}{S^3}\phi_0^4 cos2Qcos2q,$

$c_{11} = -a_1\phi_0,\ c_{12} = 0,$

$a_2 = -(\frac{4J'}{S^2} + \frac{4J'_1}{S^2})\phi_0^3 cos2q - \frac{4J'_2}{S^2}\phi_0^3 cos4q - (\frac{4J'}{S^2} + \frac{4J'_1}{S^2} + \frac{4J'_2}{S^2})\phi_0^3 - (\frac{12J'}{S^2} + \frac{12J'_1}{S^2})\phi_0^3 cosQ - \frac{12J'_2}{S^2}\phi_0^3 cos2Q + \frac{12A}{S^2}\phi_0^3 + (\frac{4J'}{S^3} + \frac{4J'_1}{S^3})\phi_0^5 cos2qcosQ + \frac{4J'_2}{S^3}\phi_0^5 cos2Qcos4q,$

$c_{13} = -a_2\phi_0,\ c_{14} = 0,\ c_{21} = 0,$

$c_{22} = -(\frac{4J'}{S} + \frac{4J'_1}{S})(1 - cosQ)cosq - \frac{4J'_2}{S}cos2q(1 - cos2Q) + (\frac{4J'}{S} + \frac{4J'_1}{S} + \frac{4J'_2}{S}) - (\frac{8J'}{S^2} + \frac{8J'_1}{S^2})\phi_0^2 cosQcosq - \frac{8J'_2}{S^2}\phi_0^2 cos2cos2q - (\frac{6J'}{S^2} + \frac{6J'_1}{S^2})\phi_0^2 cosQcosq - \frac{6J'_2}{S^2}\phi_0^2 cos2Qcos2q - (\frac{4J'}{S^2} + \frac{4J'_1}{S^2})\phi_0^2(1 - 2cosQ)cos2q - \frac{4J'_2}{S^2}\phi_0^2(1 - 2cos2Q)cos4q + (\frac{4J'}{S^2} + \frac{4J'_1}{S^2} + \frac{4J'_2}{S^2})\phi_0^2 + (\frac{12J'}{S^2} + \frac{12J'_1}{S^2})\phi_0^2 cosq + \frac{12J'_2}{S^2}\phi_0^2 cos2q - \frac{12A}{S^2} - (\frac{4J'}{S^3} + \frac{4J'_1}{S^3})\phi_0^4 cos2q - \frac{4J'_2}{S^3}\phi_0^4 cos4q - \frac{24A}{S^3}\phi_0^4,$

$c_{23} = 0,$

$c_{24} = (\frac{6J'}{S^2} + \frac{6J'_1}{S^2})\phi_0^2 cosQcosq + \frac{6J'_2}{S^2}\phi_0^2 cos2Qcos2q - (\frac{4J'}{S^2} + \frac{4J'_1}{S^2})\phi_0^2 cos2q - \frac{4J'_2}{S^2}\phi_0^2 cos4q - (\frac{4J'}{S^2} + \frac{4J'_1}{S^2} + \frac{4J'_2}{S^2})\phi_0^2 - (\frac{12J'}{S^2} + \frac{12J'_1}{S^2})\phi_0^2 cosq - \frac{12J'_2}{S^2}\phi_0^2 cos2q - \frac{12A}{S^2} + (\frac{4J'}{S^3} + \frac{4J'_1}{S^3})\phi_0^4 cosQcos2q + \frac{4J'_2}{S^3}\phi_0^4 cos2Qcos4q - \frac{24A}{S^3}\phi_0^4,$

$c_{31} = c_{13},\ c_{32} = 0,\ c_{33} = c_{11},\ c_{34} = 0,$

$c_{41} = 0,\ c_{42} = c_{24},\ c_{43} = 0,\ c_{44} = c_{22},$

The coefficients of Eq.(58) are given by

$l_0 = 2(c_{22}c_{11} + c_{13}c_{24}),$

$l_1 = (c_{22}c_{11})^2 + (c_{13}c_{24})^2 - (c_{11}c_{24})^2,$

These coefficients allow us to express

$\alpha_1 = \frac{-l_0 - \sqrt{l_0^2 - 4l_1}}{2\phi_0^2}$, $\alpha_2 = \frac{-l_0 + \sqrt{l_0^2 - 4l_1}}{2\phi_0^2}$ and

$$\Omega_{im1} = \pm\sqrt{\frac{-l_0 - \sqrt{l_0^2 - 4l_1}}{2\phi_0^2}},\ \Omega_{im2} = \pm\sqrt{\frac{-l_0 + \sqrt{l_0^2 - 4l_1}}{2\phi_0^2}}.$$

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