# Einstein's Gravitational Field Equations in integral form via Bianchi Identities

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***Abstract***: The purpose of the paper is to obtain the integral form of the Gravitational Field Equations. In fact, we have shown that the Gravitational Field across the surface S bounding the region $V$ is zero.

***Keywords***: *Bianchi Identities, Einstein's Gravitational Field Equations, Riemannian geometry.*

## 1. Introduction

In Classical Differential Geometry, second identity (1.2) is called the second Bianchi identity proved by Bianchi in 1889. Many physicists and mathematician established connections with the Bianchi identities. J. B. Davies [4] used the Bianchi identities to find curvature torsion relations and contribution of symmetric curvature to the gravitational field. S. M. Bhati, G. Murali, G, Deepa, and Ch. Sanjay [1] obtain connection between continuity equations in Fluid Dynamics and Bianchi identities. It is shown that fluid flux across the closed surface S bounding volume V is zero. Recently, S. M. Bhati, G. Murali, and Ch. Sanjay [2, 3], obtain the law of electric field and magnetic field in the space time system with respect to any frame of reference using Bianchi identities.

Rowe David [5] has proved the Einstein's Gravitational Field Equations and energy-momentum conservation laws in the space time system. He has shown in his article that how the Gravitational Field Equations are significantly related to the Bianchi identities of Riemannian geometry. Einstein found the Gravitational Field Equations in differential form and these equations are locally expressed at each point in space. Concurrent research avenues explored by the studies which were authored by Murali Etc.al [8]-[25] delved into different forms, providing substantial insight into the nature of the work reported.

Let $M^n$ be a connected differential manifold of dimension $n > 2$ covered by system of co-ordinate neighbourhood $(U; x^\lambda)$, where $U$ is the neighbourhood and $x^\lambda$ denote the local co-ordinates in $U$ and the indices $\lambda, \mu, \nu, \kappa, \ldots\ldots$ taking on the values $1,2,3, \ldots, n$. Let $g$ be the Riemannian metric which is the second order tensor with covariant components $g_{\lambda\mu}$ and with contravariant components $g^{\lambda\mu}$.

Let $\nabla$ be the Riemannian connection with components $\Gamma_{\lambda\nu}{}^{\eta}$ , called Riemann Christoffel symbols. Raising and lowering of indices are carried out using $g_{\lambda\nu}$ and $g^{\lambda\nu}$ respectively. Einstein summation conventions are used in this paper.Let $R_{\mu\nu\kappa}{}^{\sigma}$ and $R_{\lambda\nu}$ be the Riemannian Christoffel curvature tensor field of type $(1,3)$ and Ricci curvature tensor of $M^n$ respectively. Let $r$ be the scalar curvature of space time system, that is, $(M^3, t)$, that is, $M^4$. We quote the following two famous identities from the Differential Geometry which are needed in our study [7]

$$R_{\kappa\nu\mu}{}^{\sigma} = -R_{\nu\kappa\mu}{}^{\lambda}, \qquad (1.1)$$

$\nabla_\sigma R_{\kappa\nu\mu}{}^{\lambda} + \nabla_\kappa R_{\nu\sigma\mu}{}^{\lambda} + \nabla_\nu R_{\sigma\kappa\mu}{}^{\lambda} = 0$ (1.2)

## 2.Fomulation

In this section, we consider the 4-dimensional space $(M^3, t) = M^4$. The indices $\lambda, \mu, \nu, \eta$ running over the range 1, 2, 3, 4. One of the four co-ordinates may be taken as time co-ordinate t. Einstein [6] studied the gravitational field equations with respect to the theory of relativity. David[5] has proved the Einstein's Gravitational Field Equations and energy-momentum conservation laws in the space time system. We study our results in space-time system.

Using the Bianchi identities (1.1), (1.2), contracting Bianchi identity (1.2) with respect to $\lambda$ and $\sigma$, we get

$$\nabla_\eta R_{\kappa\nu\mu}{}^{\eta} - \nabla_\kappa R_{\nu\mu} + \nabla_\nu R_{\kappa\mu} = 0$$

Now multiplying by $g^{\nu\mu}$, we get

$$\nabla_\sigma R_\kappa{}^{\sigma} = \frac{1}{2}\nabla_\kappa R$$

Again, multiplying by $g^{\lambda\kappa}$, we have

$\nabla_\sigma\left(g^{\lambda\kappa} R_\kappa{}^{\sigma}\right) = \frac{1}{2} g^{\lambda\kappa}\nabla_\kappa R$ (2.1)

$\nabla_\sigma R^{\lambda\sigma} = \frac{1}{2}\nabla_\lambda R$ (2.2)

Where $\nabla^\lambda = g^{\lambda\kappa}\nabla_\kappa$. einstein's gravitational tensor field $G^{\lambda\nu}$ of the type $(0,2)$ on $M^4$ is defined by [5]

$$G^{\lambda\nu} = R^{\lambda\mu} - \frac{1}{2} r g^{\lambda\nu}$$

We quote the Stoke's Theorem which is needed in this section.

## 3. Methods

**Stoke's Theorem:** If $G$ be the tensor field of type $(2,0)$ on $M^4$, then

$\int_{M^4} \nabla_\nu G^{\lambda\nu}\, dV = \int_S G^{\lambda\nu} N_\nu\, dS$ (2.3)

where $N_\nu$ is the component of the unit outward normal to the boundary, $G^{\lambda\nu}$ is the contravariant components of tensor $G$, $dS$ is the surface element of $S$ and $dV$ is the volume element of $M^4$

Einstein established that if $G$ represents the gravitational field tensor, then the divergence of $G$ is zero, that is, gravitational field equations in the differential form are given by [6]

$$\nabla_\lambda G^{\nu\lambda} = 0$$

This is locally expressed at each point in space.

The main purpose of the paper is to obtain the gravitational field equations globally, that is, in the integral form. In this connection, we prove the following Theorem.

**Theorem 2.1**. *Gravitational Field Equations in the Integral Form: If $G$ be the tensor field of type $(2,0)$ in $M^4$, then the Gravitational Field Equations in the integral form is given by*

$\int_S G^{\lambda\nu} N_\nu\, dS = 0$ (2.4)

*$N_\nu$ is the component of the unit outward normal to the boundary, $G^{\lambda\nu}$ is the contravariant component of the tensor field $G$, $dS$ is the surface element of $S$.*

*Proof.* From equation (2.3)

$$\int_S G^{\lambda\nu} N_\nu\, dS = \int_{M^4} \nabla_\nu G^{\lambda\nu}\, dV$$

$$= \int_{M^4} \nabla_\nu \left( R^{\mu\nu} - \frac{1}{2} r g^{\mu\nu} \right) dV$$

$$= \int_{M^4} \left[ \nabla_\nu R^{\mu\nu} - \frac{1}{2} \nabla_\nu r g^{\mu\nu} \right] dV$$

$$= \int_{M^4} \left[ \frac{1}{2} \nabla^\mu r - \frac{1}{2} \nabla^\mu r \right] dV$$

$$= 0,$$

Where in we have used (2.1) and (2.2). This proves (2.4).

**4. Results**

Einstein established that if $G$ represents the gravitational field tensor, then the divergence of $G$ is zero .The equation (2.4) simply asserts that the gravitational field across the closed surface S bounding the region V is zero in the space time system with respect to any frame of reference.

**5. Conclusions**

The integral form of the Gravitational Field Equations are obtained using bianchi identities .we have also shown that the Gravitational Field across the surface S bounding the region *V* is zero.

**References**

[1] S. M. Bhati, G. Murali, G. Deepa and Ch. Sanjay, Continuity equations in Fluid Dynamics and Bianchi identities, *Strad Research*,**8**(5) (2021), 587-591.

[2] S. M. Bhati, G. Murali, and Ch. Sanjay, Law of Electric Field via Bianchi identities,*Journal of Applied Science and Engineering*,**25**(4) (2022) 669-662.

[3] S. M. Bhati, G. Murali, and Ch. Sanjay, Law of Magnetic Field via Bianchi identities, Journal of Propulsion technology, 44(5) (2023).

[4] J. B. Davies,New Curvature-Torsion Relations Through Decomposition of the Bianchi Identities,*Foundation of Physics*,**18**(5) (1988), 563-569.

[5] Rowe E. David, Years Ago, *Mathematical Intelligencer,Springer, New York*, **24**(2002), 59-64.

[6] A. Einstein, Prinziielles zur verallgem ener ten R Relati vita tsth eores und gravitationstheori, *Physik aliche Zeitschnift,***15**(1914), 176-578.

[7] K. Yano, Formulas in Riemannian Geometry, *Marcel Dekker, New York*, (1970),13-14

[8] Murali Gundagani, MCK Reddy,Sivaiah (2012) Finite element solution of thermal radiation effect on unsteady MHD flow past a vertical porous plate with variable suction *American Academic & Scholarly Research Journal* 4 (3), 3-22,2012.

[9] MCK Reddy, G Murali, S Sivaiah, NVN Babu, Heat and mass transfer effects on unsteady MHD free convection flow past a vertical permeable moving plate with radiation, *International Journal of Applied Mathematical Research*, 1 (2) (2012),189-205.

[10] G Murali, NVN Babu, Effect of Radiation on MHD Convection Flow Past a Vertical Permeable Moving Plate, *International Journal of Advances in Applied Sciences*, Vol.1(1), March 2012, pp. 19~28.

[11] S.Sivaaih,G.Murali, MCK Reddy,R.Srinivasa Raju,Unsteady MHD mixed convection flow past a vertical porous plate in prsesence of radiation, *International journal of basic and applied sciences*, 1(4),651-666,2012.

[12] Sivaiah Sheri, Murali Gundagani. Maruthi Prasad Karanamu, Analysis of Heat and Mass Transfer Effects on an Isothermal Vertical Oscillating Plate, *Walailak Journal of science and technology,*9(4): 407-415,2012.

[13] S Sivaiah, G Murali, MCK Reddy,Finite Element Analysis of Chemical Reaction and Radiation Effects on Isothermal Vertical Oscillating Plate with Variable Mass Diffusion, *International Scholarly Research Network ISRN Mathematical Physics* Volume 2012, Article ID 401515, doi:10.5402/2012/401515.

[14] Murali Gundagani, Sivaiah Sheri, Paul Ajit, Marpadaga Chenna Krishna Reddy,Radiation Effects on an unsteady MHD convective flow past a semi-infinite vertical permeable moving plate embedded in a porous medium with viscous dissipation, *Walailak Journal of science and technology*,10(5): 499-515,2013.

[15] Deepa Gadipally,Murali Gundagani,Analysis of soret and dufour effects on unsteady MHD flow past a semi infinite vertical porous plate via finite difference method, *International journal of applied physics and mathematics*,4(5),332-344,2014.

[16] G Deepa, G Murali, Effects of viscous dissipation on unsteady MHD free convective flow with thermophoresis past a radiate inclined permeable plate, *Iranian Journal of Science and Technology (Sciences)*, 38A3, 2014.

[17] G Murali, A Paul, NVN Babu, Heat and mass transfer effects on an unsteady hydromagnetic free convective flow over an infinite vertical plate embedded in a porous medium with heat absorption, *Int. J. Open Problems Compt. Math* 8 (1), 2015.

[18] NVN Babu, Ajit Paul,G Murali, Soret and Dufour effects on unsteady hydromagnetic free convective fluid flow past an infinite vertical porous plate in the presence of chemical reaction, *Journal of science and arts*, 15 (1), pp. 99-111, 2015.

[19] Murali Gundgagani, Unsteady MHD free convection viscous dissipative flow past an infinite vertical plate with constant suction and heat source/sink, *Journal of science and arts*,15(1),2015

[20] G.Murali,Ajit Paul, NVN Babu,Numerical study of chemical reaction effects on unsteady MHD fluid flow past an infinite vertical plate embedded in a porous medium with variable suction, *Electronic Journal of mathematical analysis and applications*,3(2),179-192,2015.

[21] G Murali,EM Reddy,Narepalepu Venkata N.B, Heat and mass transfer with free convection MHD flow past a vertical porous plate: Numerical study, *International journal of science and engineering*, Vol. 8(2),pp.95-103, April 2015.

[22] N.V.N Babu, G.Murali, S.M Bhati, Casson fluid performance on natural convective dissipative couette flow past an infinite vertically inclined plate filled in porous medium with heat transfer, MHD and hall current effects, *International journal of Pharmaceutical Research,* 10(4), 2018.

[23] Deepa Gadipally, Murali Gundagani, Narendra NV Babu, Effects of thermal radiation and chemical reaction on unsteady hydromagnetic mixed convection flow past an infinite vertical plate, *Journal of science and arts,* 3(44), pp.763-776,2018.

[24] Murali G, NVN Babu, Convective MHD Jeffrey Fluid Flow Due to Vertical Plates with Pulsed Fluid Suction:A Numerical Study, *Journal of computational applied mechanics* DOI *10.22059/JCAMECH.2023.351326.773,2023.*

[25] Murali Gundagani, Sivaiah Sheri,Ajit Paul,MCK Reddy,Unsteady magnetohydrodynamic free convective flow past a vertical porous plate *International Journal of Applied Science and Engineering,11(3),*pp.267-275,2013.