Einstein’s Gravitational Field Equations in integral form via Bianchi Identities

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Abstract: The purpose of the paper is to obtain the integral form of the Gravitational Field Equations. In fact, we have shown that the Gravitational Field across the surface S bounding the region V is zero.

Keywords: Bianchi Identities, Einstein’s Gravitational Field Equations, Riemannian geometry.

1. Introduction

In Classical Differential Geometry, second identity (1.2) is called the second Bianchi identity proved by Bianchi in 1889. Many physicists and mathematician established connections with the Bianchi identities. J. B. Davies [4] used the Bianchi identities to find curvature torsion relations and contribution of symmetric curvature to the gravitational field. S. M. Bhati, G. Murali, G, Deepa, and Ch. Sanjay [1] obtain connection between continuity equations in Fluid Dynamics and Bianchi identities. It is shown that fluid flux across the closed surface S bounding volume V is zero. Recently, S. M. Bhati, G. Murali, and Ch. Sanjay [2, 3], obtain the law of electric field and magnetic field in the space time system with respect to any frame of reference using Bianchi identities.

Rowe David [5] has proved the Einstein’s Gravitational Field Equations and energy-momentum conservation laws in the space time system. He has shown in his article that how the Gravitational Field Equations are significantly related to the Bianchi identities of Riemannian geometry. Einstein found the Gravitational Field Equations in differential form and these equations are locally expressed at each point in space. Concurrent research avenues explored by the studies which were authored by Murali Etc.al [8]-[25] delved into different forms, providing substantial insight into the nature of the work reported.

Let be a connected differential manifold of dimension covered by system of co-ordinate neighbourhood \( U; x^k \), where \( U \) is the neighbourhood and \( x^k \) denote the local co-ordinates in \( U \) and the indices \( \lambda, \mu, \nu, \kappa, \ldots \) taking on the values \( 1, 2, 3, \ldots, n \). Let \( g \) be the Riemannian metric which is the second order tensor with covariant components \( g_{\lambda\mu} \) and with contravariant components \( g^{\lambda\mu} \).

Let \( \nabla \) be the Riemannian connection with components \( \Gamma^\eta_{\lambda\nu} \), called Riemann Christoffel symbols. Raising and lowering of indices are carried out using \( g_{\lambda\nu} \) and \( g^{\lambda\nu} \) respectively. Einstein summation conventions are used in this paper. Let \( R_{\mu\nu\sigma}^{\rho} \) and \( R_{\lambda\nu} \) be the Riemannian Christoffel curvature tensor field of type \((1, 3)\) and Ricci curvature tensor of \( M^n \) respectively. Let \( r \) be the scalar curvature of space time system, that is, \((M^4, t)\), that is, \( M^4 \). We quote the following two famous identities from the Differential Geometry which are needed in our study [7]

\[ R_{\kappa\nu\mu}^{\lambda} = -R_{\kappa\mu\nu}^{\lambda}, \]  

(1.1)
\[ \nabla_{\sigma} R_{\kappa\nu\eta}^{\lambda} + \nabla_{\kappa} R_{\nu\eta\mu}^{\lambda} + \nabla_{\nu} R_{\kappa\lambda\mu} = 0 \quad (1.2) \]

2. Formulation

In this section, we consider the 4-dimensional space \((M^3, t) = M^4\). The indices \(\lambda, \mu, \nu, \eta\) running over the range 1, 2, 3, 4. One of the four co-ordinates may be taken as time co-ordinate \(t\). Einstein [6] studied the gravitational field equations with respect to the theory of relativity. David [5] has proved the Einstein’s Gravitational Field Equations and energy-momentum conservation laws in the space time system. We study our results in space-time system.

Using the Bianchi identities (1.1), (1.2), contracting Bianchi identity (1.2) with respect to \(\lambda\) and \(\sigma\), we get

\[ \nabla_{\eta} R_{\kappa\nu\eta\mu}^{\lambda} - \nabla_{\nu} R_{\kappa\lambda\mu}^{\lambda} + \nabla_{\mu} R_{\kappa\nu\lambda} = 0 \]

Now multiplying by \(g^{\nu\mu}\), we get

\[ \nabla_{\sigma} R_{\kappa}^{\lambda \sigma} = \frac{1}{2} \nabla_{\kappa} R \quad (2.1) \]

Again, multiplying by \(g^{\lambda\kappa}\), we have

\[ \nabla_{\sigma} \left( g^{\lambda\kappa} R_{\kappa}^{\lambda \sigma} \right) = \frac{1}{2} g^{\lambda\kappa} \nabla_{\kappa} R \quad (2.2) \]

Where \(\nabla_{\kappa} = g^{\lambda\kappa} \nabla_{\lambda}\). Einstein’s gravitational tensor field \(G^{\lambda\nu}\) of the type \((0,2)\) on \(M^4\) is defined by [5]

\[ G^{\lambda\nu} = R^{\lambda\nu} - \frac{1}{2} g^{\lambda\nu} \]

We quote the Stoke’s Theorem which is needed in this section.

3. Methods

Stoke’s Theorem: If \(G\) be the tensor field of type \((2,0)\) on \(M^4\), then

\[ \int_{M^4} \nabla_{\nu} G^{\lambda\nu} dV = \int_{S} G^{\lambda\nu} N_{\nu} dS \quad (2.3) \]

where \(N_{\nu}\) is the component of the unit outward normal to the boundary, \(G^{\lambda\nu}\) is the contravariant components of tensor \(G\), \(dS\) is the surface element of \(S\) and \(dV\) is the volume element of \(M^4\).

Einstein established that if \(G\) represents the gravitational field tensor, then the divergence of \(G\) is zero, that is, gravitational field equations in the differential form are given by [6]

\[ \nabla_{\lambda} G^{\lambda\nu} = 0 \]

This is locally expressed at each point in space.

The main purpose of the paper is to obtain the gravitational field equations globally, that is, in the integral form. In this connection, we prove the following Theorem.

Theorem 2.1. Gravitational Field Equations in the Integral Form: If \(G\) be the tensor field of type \((2,0)\) in \(M^4\), then the Gravitational Field Equations in the integral form is given by

\[ \int_{S} G^{\lambda\nu} N_{\nu} dS = 0 \quad (2.4) \]

\(N_{\nu}\) is the component of the unit outward normal to the boundary, \(G^{\lambda\nu}\) is the contravariant component of the tensor field \(G\), \(dS\) is the surface element of \(S\).

Proof. From equation (2.3)

\[ \int_{S} G^{\lambda\nu} N_{\nu} dS = \int_{M^4} \nabla_{\nu} G^{\lambda\nu} dV \]
\[
\begin{align*}
\frac{1}{2} \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \right) & = \int_\mathcal{M} \nabla_\nu \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \right) dV \\
& = \int_\mathcal{M} \left[ \nabla_\nu R_{\mu \nu} - \frac{1}{2} \nabla_\nu g_{\mu \nu} \right] dV \\
& = \int_\mathcal{M} \left[ \frac{1}{2} \nabla_\mu R_{\mu} - \frac{1}{2} \nabla_\mu r \right] dV \\
& = 0,
\end{align*}
\]

Where in we have used (2.1) and (2.2). This proves (2.4).

4. Results

Einstein established that if \( G \) represents the gravitational field tensor, then the divergence of \( G \) is zero. The equation (2.4) simply asserts that the gravitational field across the closed surface \( S \) bounding the region \( V \) is zero in the space time system with respect to any frame of reference.

5. Conclusions

The integral form of the Gravitational Field Equations are obtained using bianchi identities. We have also shown that the Gravitational Field across the surface \( S \) bounding the region \( V \) is zero.

References


