Monophonic Polynomial of the cartesian product of some graphs

P.Arul Paul Sudhahar [1] , W.Jebi [2]

[1]Assistant Professor, Department of Mathematics Government Arts and Science College, Nagercoil 629004, Tamilnadu.

[2]Research Scholar (Register No: 20111172092015),
Department of Mathematics, Rani Anna Government college for women,
Tirunelveli - 627008,
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627012, Tamilnadu, India.

Abstract

Let $M(G,i)$ be the family of monophonic sets of a graph G with cardinality i and let $m(G,i) = |M(G,i)|$. Then the monophonic polynomial $M(G,x)$ of G is defined as $M(G,x) = \sum_{i=m(G)}^{n} m(G,i)x^i$, where $m(G)$ is the monophonic number of G. In this paper we have determined the sufficient condition for the monophonic set of $G \square K_n$. Also, we have calculated the monophonic polynomial of the cartesian product of some specific graphs by generating function method.

Keywords: Monophonic set, Cartesian product, Monophonic number.

AMS Subject Classification: 05C12.

I. Introduction

The graph G considered in this paper is finite, simple, undirected and connected with vertex set V (G) and edge set E(G) respectively. The order and size of G is denoted by n and m respectively. [1, 2] is referred for basic graph theoretic definitions. The distance $d(u,v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. The neighborhood $N(v)$ of a vertex v is the set of all vertices adjacent to v. For any subset S of V(G), the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with the vertex set S. A clique of G is a complete subgraph of G. A clique of G is said to be maximal if it is not properly contained in another clique of G. The order of a maximum clique of G is the clique number of G and is denoted by $\omega(G)$. A vertex v is said to be an extreme vertex if the subgraph $\langle N(v) \rangle$ is complete. Two vertices (x_1, y_1) and (x_2, y_2) in the cartesian product $G \square H$ are adjacent if and only if either $x_1 = x_2$ and y_1 is adjacent to y_2 or $y_1 = y_2$ and x_1 is adjacent to x_2. The study of monophonic number of a graph was initiated by Pelayo et al. in [3]. Any chordless path connecting the vertices u and v is called a $u - v$ monophonic path. The monophonic closure of a subset S of V (G) is defined by $J_G[S] = \cup_{u,v \in S} J_G[u,v]$, where $J_G[u,v]$ is the set containing the vertices u,v and all vertices lying on some $u - v$ monophonic path. If $J_G[S] = V(G)$, then S is said to be a monophonic set in G. A monophonic set in G of least order is called a minimum monophonic set of G. The order of the minimum monophonic set of G is the monophonic number of G and is denoted by $m(G)$. The domination polynomial of a graph was introduced by Arocha and Llano in [4] and was later studied by S. Alikhani and Y. H. Peng in [5]. Domination polynomial of graph products are studied by Kotek et al. in [6]. We developed an idea to work on the monophonic polynomial of graph products after reading this paper. The study of monophonic polynomial of a graph was initiated by Arul Paul Sudhahar et al. in [7].

II. Preliminary Results

Theorem 2.1. [8] Let G and H be connected graphs such that G is non complete and $m(G) = 2$. Then $m(G \square H) = 2$.

Theorem 2.2. [8] Let G and H be non complete connected graphs. Then $m(G\square H) = 2$.

Theorem 2.3. [8] If the graph G is a tree, then $m(G\square K_n) = m(G)$.

III. Characterisation of monophonic set of cartesian product

Theorem 3.1. Let G be a graph. If $L = (u_1, v_1), (u_2, v_2), \dots, (u_k, v_k)$ is a monophonic path in $G\square K_n$, then the following holds

1. If all u_i's are distinct and v_i's are equal, then $(u_1, u_2, \dots, u_k)$ is a monophonic path in G.
2. If all u_i's are equal and v_i's are distinct, then $k = 2$.

Proof. 1. Since all the u_i's are distinct and v_i's are equal, $d_{G\square K_n}((u_i, v_i), (u_{i+1}, v_{i+1})) = 1$ for all $1 \leq i \leq k-1$. Therefore, $d_G(u_i, u_{i+1}) = 1$ for all $1 \leq i \leq k-1$. Also, $d_{G\square K_n}((u_i, v_i), (u_j, v_j)) \geq 2$ for all $j \neq i+1$ and $1 \leq i < j \leq k$, it follows that $d_G(u_i, u_j) \geq 2$ for all $j \neq i+1$ and $1 \leq i < j \leq k$. Hence $(u_1, u_2, \dots, u_k)$ is a monophonic path in G.

2. Since all u_i's are equal, all the vertices in the monophonic path L is from the same copy of K_n. Suppose $k \geq 3$. Then the subgraph $\langle (u_i, v_i), (u_j, v_j), (u_l, v_l) \rangle$, $i, j, l \in \{1, 2, \dots, k\}$ and $i \neq j \neq l$ form a complete graph K_3, a contradiction. Hence $k \leq 2$. Since any monophonic path contains at least two vertices, it can be easily verified that $k = 2$.

Theorem 3.2. $S \subseteq V(K_p\square K_q)$ is a monophonic set of $K_p\square K_q$ if and only if there exist at least one vertex from at least two distinct copies of K_p or K_q.

Proof. Let $S \subseteq V(K_p\square K_q)$ be a monophonic set of $K_p\square K_q$. Suppose S contains vertices from a single copy of K_q or K_p. Then $J_{K_p\square K_q}[S] = S \neq V(K_p\square K_q)$. This implies that S is not a monophonic set of $K_p\square K_q$, contradiction to our assumption. Hence S contains at least one vertex from at least two distinct copies of K_p or K_q.

Conversely, assume that there exists at least one vertex from at least two distinct copies of K_p. Let $S_1, S_2, \dots, S_q$ be the q distinct copies of K_p in $K_p\square K_q$. Let $(x, y_i) \in S \cap S_i$ and $(z, y_j) \in S \cap S_j$, where $i \neq j$ and $xz \notin E(K_p)$, $y_i y_j \notin E(K_q)$. Let $(a, b) \in V(K_p\square K_q)$. If $a = x$, then $(x, y_i), (a, b), (z, b), (z, y_j)$ will be a monophonic path joining (x, y_i) and (z, y_j). If $b = y_i$, then $(x, y_i), (a, b), (a, y_j), (z, y_j)$ will be a monophonic path joining (x, y_i) and (z, y_j). Similarly, as above we can able to find a monophonic path joining (x, y_i) and (z, y_j) for the cases $a = z$ and $b = y_j$. If $a \neq x, z$ and $b \neq y_i, y_j$, then $(x, y_i), (a, y_i), (a, b), (z, b), (z, y_j)$ will be the monophonic path joining (x, y_i) and (z, y_j). Thus $(a, b) \in J_{K_pK_q}[(x, y_i), (z, y_j)]$. Hence S forms a monophonic set of $K_p\square K_q$.

Corollary 3.3. For the complete graphs K_p and K_q, $m(K_p\square K_q) = 2$.

Proof. Let $S_1, S_2, \dots, S_q$ be the q distinct copies of K_p in $K_p\square K_q$. By Theorem 3.2, it follows that the monophonic path connecting two non adjacent vertices from two distinct S_i's form a monophonic set of $K_p\square K_q$. Therefore, $m(K_p\square K_q) \leq 2$. Since $m(G) \geq 2$ for any connected graph G, we get $m(K_p\square K_q) = 2$.

Theorem 3.4. Let G be a non complete connected graph of order n. If $S \subseteq V(G)$ is a minimum monophonic set of G then $S \times \{v_i\}$, $1 \leq i \leq p$ is a monophonic set of $G\square K_p$ for each i.

Proof. Let $G_1, G_2, \dots, G_p$ be the p copies of G and $V(K_p) = \{v_1, v_2, \dots, v_p\}$. Let $S \subseteq V(G)$ be a minimum monophonic set of G and $(x, y) \in V(G\square K_p)$. Suppose $x \in V(G)\backslash S$. Since S is monophonic in G, there exists a monophonic path $u_1, u_2, \dots, u_k$ such that $u_1, u_k \in S$, $x = u_j$ for some $1 < j < k$.

Therefore, $(u_1, v_i), (u_2, v_i), \ldots, (u_k, v_i)$ for $1 \leq i \leq k$ will be a monophonic path in $G\square K_p$. If $y \in G_i$, then it will be covered by the monophonic path connecting (u_1, v_i) and (u_k, v_i). Now, let us consider the case when $y \notin G_i$. Let $y \in G_l$ and $i \neq l, 1 \leq l \leq p$. $(u_1, v_i), (u_1, v_l), (u_2, v_l), \ldots, (u_k, v_l), (u_k, v_i)$ will be a monophonic path in $G\square K_p$. Hence $(x, y) \in J_{G\square H}[(u_1, v_i), (u_k, v_i)]$. Since $u_1, u_k \in S$ is arbitrary, it can be shown that $(x, y) \in J_{G\square K_p}[S \times \{v_i\}]$ for $1 \leq i \leq p$. Hence $S \times \{v_i\}$ is a monophonic set of $G\square K_p$ for every $1 \leq i \leq p$.

Corollary 3.5. For the connected graph G with $n \geq 2$, $m(G\square K_p) \leq m(G)$.

Proof. By Theorem 3.4 we have, $m(G\square K_p) \leq |S| \leq m(G)$.

Remark 3.6. The converse of Theorem 3.4 is not true. For the graph G given in Figure 3.3, $F = \{(u_4, v_1), (u_5, v_1), (u_6, v_1)\}$ forms the monophonic set of $G\square K_2$. But $\{u_4, u_5, u_6\}$ is not a monophonic set of the graph G given in Figure 3.1.

Remark 3.7. The converse of Theorem 3.4 may not be true even for the cartesian product of two non complete graphs G and H. For the graph $G\square H$ given in Figure 3.6, $K = \{(u_1, v_1), (u_4, v_1)\}$ forms the monophonic set of $G\square H$. But $L = \{u_1, u_4\}$ is not a monophonic set of the graph G given in Figure 3.4.

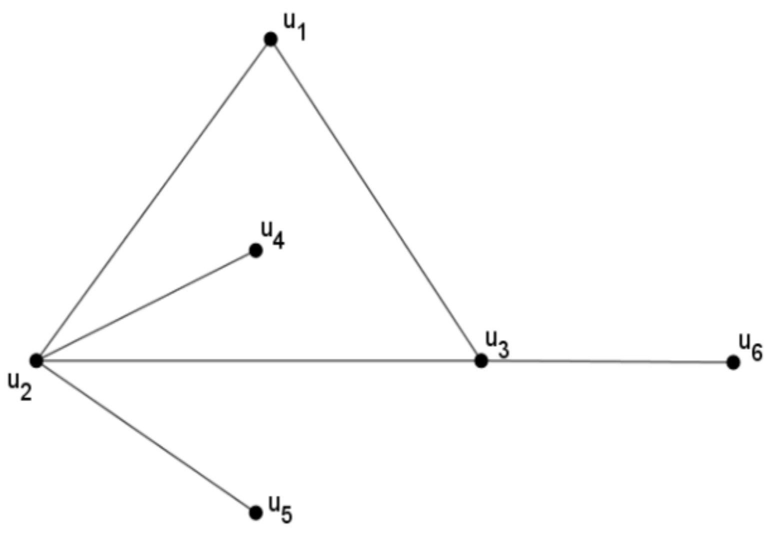

Figure 3.1 Graph G

Figure 3.2 Complete graph K_2

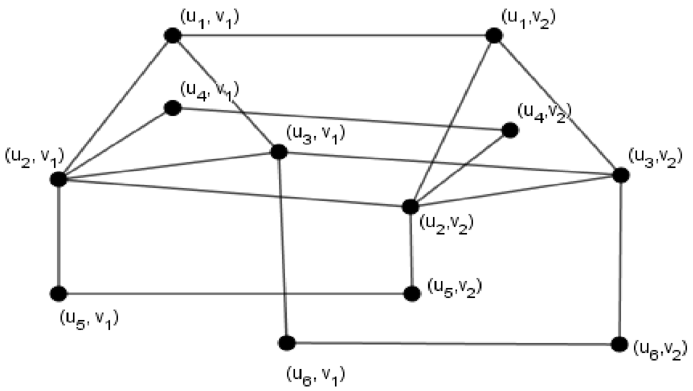

Figure 3.3 G □H

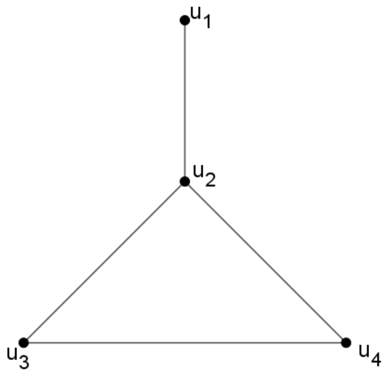

Figure 3.4 Graph G

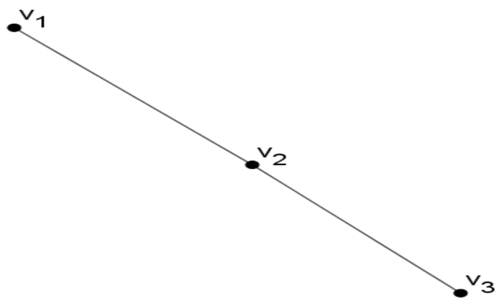

Figure 3.5: Path P_3

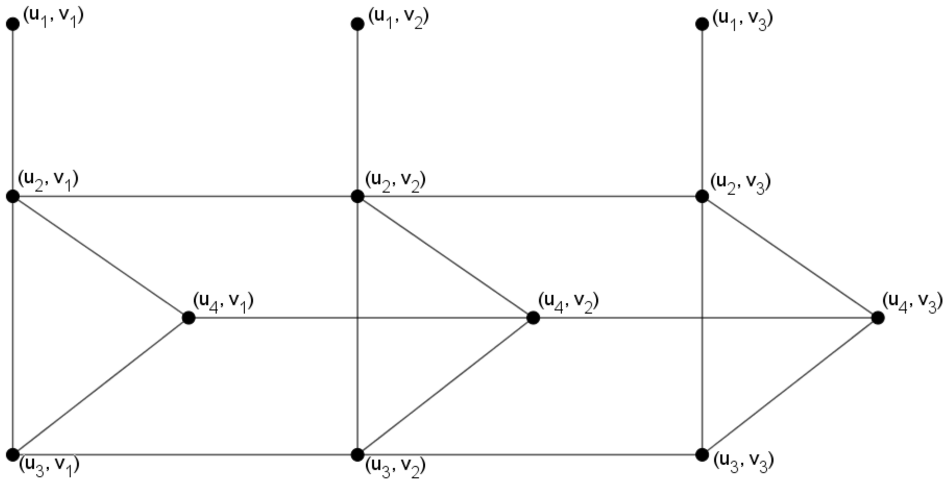

Figure 3.6 G □ H

IV. Monophonic polynomial of the cartesian product of some graphs

Theorem 4.1. For two complete graphs K_n and K_m, $M(K_n \square K_m, x) = [(1 + x)^{nm} - (1 - nmx)] - n[(1 + x)^m - 1] - m[(1 + x)^n - 1]$.

Proof. By Corollary 3.3, we have $m(K_n \square K_m) = 2$. Any two non adjacent vertices of $K_n \square K_m$ forms the monophonic set of cardinality 2. The generating function to choose at least two vertices of $K_n \square K_m$ is $[(1 + x)^{nm} - (1 + nmx)]$. From this generating function we have to neglect the case in which all the vertices are from same copy of K_n or K_m. The generating function to choose at least two vertices from a copy of K_n is $(1 + x)^n - (1 + nx)$. Since we have m copies of K_n, the generating function to choose vertices from the m copy of K_n is given by $m[(1 + x)^n - (1 + nx)]$. Similarly, the generating function for selecting vertices from the n copies of K_m is $n[(1 + x)^m - (1 + mx)]$. Hence the monophonic polynomial of $K_n \square K_m$ is given by

$$M(K_n \square K_m, x) = [(1 + x)^{nm} - (1 + nmx)] - n[(1 + x)^m - (1 + mx)]$$
$$- m[(1 + x)^n - (1 + nx)]$$
$$= [(1 + x)^{nm} - (1 - nmx)] - n[(1 + x)^m - 1] - m[(1 + x)^n - 1].$$

Theorem 4.2. For the complete graph K_m and path graph P_n, $M(K_m \square P_n, x) = [(1 + x)^m - 1]^2 [1 + x]^{nm-2m}$.

Proof. Let $D_1, D_2, \dots, D_n$ be the n copies of K_m. By Theorem 2.3, we have $m(K_m \square P_n) = 2$. Any monophonic set of $K_m \square P_n$ should contain at least one vertex from both D_1 and D_n and the generating function for choosing at least one vertex from both D_1 and D_n is $[(1 + x)^m - 1]^2$. The generating function to choose vertices from D_i, $2 \le i \le n - 1$ is $[1 + x]^{nm-2m}$. Thus, the monophonic polynomial of $P_n \square P_m$ is given by $[(1 + x)^m - 1]^2 [1 + x]^{nm-2m}$.

Theorem 4.3. For the complete graph K_m and cycle C_n, $M(K_m \square C_n, x) = (1 + x)^{nm} - [1 + n(mx^2 + (1 + x)^m - 1)]$.

Proof. By Theorem 2.1, we have $m(K_m \square C_n) = 2$. Any two non adjacent vertices form the monophonic set of cardinality 2. The generating function for selecting at least two vertices of $K_m \square C_n$ is $(1 + x)^{nm} - (1 + nmx)$. From this generating function we have to subtract the case in which the vertices adjacent to each other is chosen. Any collection of vertices from a copy of K_m is adjacent to each other. The generating function for choosing at least two vertices from a copy of K_m is $(1 + x)^m - (1 + mx)$. Since there exist n copy of K_m, the generating function for selecting vertices which are adjacent to each other is $n[(1 + x)^m - (1 + mx)]$. Also, there are n ways to choose two adjacent vertices from a copy of C_n. Thus, we get nm ways to select two adjacent vertices from m copies of C_n. Since $\omega(C_n) = 2$, it is not possible to get $k > 2$ vertices which are adjacent to each other from a copy of C_n. Hence the function which generates monophonic polynomial of $K_m \square C_n$ is given by

$$M(K_m \square P_n, x) = (1 + x)^{nm} - (1 + nmx) - nmx^2 - n[(1 + x)^m - (1 + mx)]$$
$$= (1 + x)^{nm} - [1 + n(mx^2 + (1 + x)^m - 1)].$$

Theorem 4.4. For two path graphs P_n and P_m with $n \leq m$, $M(P_n \square P_m, x) = [(1 + x)^n - 1]^2[1 + x]^{nm-2n} + [(1 + x)^m - 1]^2(1 + x)^{nm-(2n+2m-4)}$.

Proof. Let $A_1, A_2, \ldots, A_m$ be the m copies of P_n in $P_n \square P_m$ respectively. Let $B_1, B_2, \ldots, B_n$ be the n copies of P_m in $P_n \square P_m$. By Theorem 2.2, we have $m(P_n \square P_m) = 2$. Any monophonic set of $P_n \square P_m$ should contain at least one vertex from both A_1 and A_m. Hence the generating function for choosing at least one vertex from both A_1 and A_m is $[(1 + x)^n - 1]^2$. The generating function to choose at least one vertex from both B_1 and B_n is $[(1 + x)^m - 1]^2$. Thus, the monophonic polynomial of $P_n \square P_m$ is $[(1 + x)^n - 1]^2[1 + x]^{nm-2n} + [(1 + x)^m - 1]^2(1 + x)^{nm-(2n+2m-4)}$.

Theorem 4.5. For the Cycles C_n and C_m, $M(C_n \square C_m, x) = (1 + x)^{nm} - (1 + nmx + 2nmx^2)$.

Proof. By Theorem 2.2 we have $m(C_n \square C_m) = 2$. Any two non adjacent vertices form the monophonic set of cardinality 2. Now we have to find the number of possible ways to select two non adjacent vertices of cardinality 2. There exists m ways for a copy of C_m to choose adjacent vertices and n ways for a copy of C_n to choose adjacent vertices. Hence, we have 2nm ways to choose two adjacent vertices in $C_n \square C_m$. These 2nm vertices have to be subtracted from the generating function to get the number of monophonic sets of cardinality 2. Since $\omega(C_n \square C_m) = 2$, for the monophonic set of cardinality $k > 2$ we have $\binom{nm}{k}$ choices. Hence the monophonic polynomial $M(C_n \square C_m, x)$ is given by $(1 + x)^{nm} - (1 + nmx + 2nmx^2)$.

Theorem 4.6. For the path graph P_n and cycle C_m, $M(P_n \square C_m, x) = (1 + x)^{nm} - [1 + nmx + (2nm - m)x^2]$.

Proof. By Theorem 2.2, we have $m(P_n \square C_m) = 2$. Any two non adjacent vertices may form a monophonic set of cardinality 2. There are m ways to choose two adjacent vertices from a copy of C_m and $n - 1$ ways to choose adjacent vertices from a copy of P_n. Hence we have $2nm - m$ ways to choose two vertices which are adjacent in $P_n \square C_m$. Thus, there are $\binom{nm}{k} - (2nm - m)$ ways to choose the monophonic set of cardinality 2. Since the induced subgraph of order $k > 2$ is non complete, the number of monophonic sets of cardinality $k > 2$ is $\binom{nm}{k}$. Hence the monophonic polynomial $M(P_n \square C_m, x)$ is $(1 + x)^{nm} - [1 + nmx + (2nm - m)x^2]$.

Theorem 4.7. For the star graph $K_{1,n}$ and complete graph K_m, $M(K_{1,n} \square K_m, x) = [(1 + x)^m - 1]^n(1 + x)^m$.

Proof. Let $\{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and $\{u_1, u_2, \ldots, u_m\}$ be the vertices of $K_{1,n}$ and K_m respectively and $\deg(v_1) = n$. Let $A_1, A_2, \ldots, A_n, A_{n+1}$ be the $n + 1$ copies of K_m in $K_{1,n} \square K_m$ and let A_1 be the copy of K_m formed by the vertex v_1. The monophonic set of $K_{1,n} \square K_m$ should contain at least one vertex from each A_i, $2 \leq i \leq n + 1$ and it is generated by $[(1 + x)^m - 1]^n$. This generating function together with the generating function to choose vertices of A_1 forms the monophonic polynomial of $K_{1,n} \square K_m$ and is given by $[(1 + x)^m - 1]^n(1 + x)^m$.

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