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Comparative Analysis of Scaling Parameter Estimation for the Lindley Distribution in Wait Time Analysis: Simulated and Real Data Applications

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Abstract: This research focuses on estimating the scale parameter for the Lindley distribution using various methods, including Maximum Likelihood, least squares, and Linear Quantile-Moment. These methods were employed to identify the most effective approach for estimating the distributional parameters. To determine the optimal parameter estimation approach, a comparison was conducted among these estimators based on the mean square error (MSE) criteria, which involved different simulation experiments. The results demonstrated the superiority of Maximum Likelihood estimation across all sample sizes. The best estimator was then utilized to study and analyze the waiting time for Zanko Bank/University of Duhok.

Keywords: *Lindley distribution, Maximum likelihood, Mean Residual Life, Stress-Strength Reliability, Waiting time.*

1. Introduction

Time serves as a fundamental element for gaining a competitive advantage in organizations and institutions. However, it has been observed that these entities, including banks, often fail to fully recognize the significance of waiting time in enhancing task efficiency. At Zanko Bank / University of Duhok, customers' waiting times undergo two distinct stages within the bank's administrative operations. The initial stage encompasses the time customers spend completing the registration process for withdrawing checks accounts, while the second stage pertains to the time spent in front of the teller. This research aims to comprehensively describe and analyze the homogeneous waiting times within Zanko Bank / University of Duhok's administrative operations.

It is crucial to note that the selection of applied probabilistic journals varies depending on the nature of the investigated systems, which can range from simple systems with single-distribution communities to complex and heterogeneous ones. Researchers encounter the challenge of choosing probability distributions that accurately correspond to the behavior of random variables within these systems, including generalized and mixed distributions. The Lindley distribution, initially proposed by Lindley in 1958 [Lindley, (1958)], belongs to the category of mixed distributions and has proven to be remarkably flexible in capturing the characteristics of systems composed of complex and heterogeneous populations. Consequently, it serves as a suitable choice for representing the diverse systems encountered. The theoretical aspect of this research involves comparing estimators such as the maximum likelihood method (MLE), the least squares method (LS), and the linear quantile moment method (LQM) in estimating the scaling parameter of the mixed Lindley distribution through various simulation experiments. Subsequently, the estimator yielding the lowest mean square error is selected and utilized. On the applied side, the study aims to describe and analyze the time spent by customers at Zanko Bank/University of Duhok.

This paper is structured as follows. In Section 2, the Lindley Distribution is examined. Section 3 explores Different Estimation Approaches for the Lindley distribution. Section 4 presents the Application of these techniques. Lastly, Section 5 provides a summary and conclusion.

2. Methodology

The probability density function (pdf) of one parameter Lindley distribution is given by[Bakouch *et al*, (2012), Ghitany *et al*, (2008)]:

$$f(x;\ \theta) = \frac{\theta^2}{(1+\theta)}(1\ +\ x)e^{-\theta x},\ x > 0,\ \theta > 0 \quad (1)$$

Figure (1) shows the (pdf) of the Lindley Function with different values of parameters $\theta = 0.5, 1, 1.5, 2, 2.5, 3$.

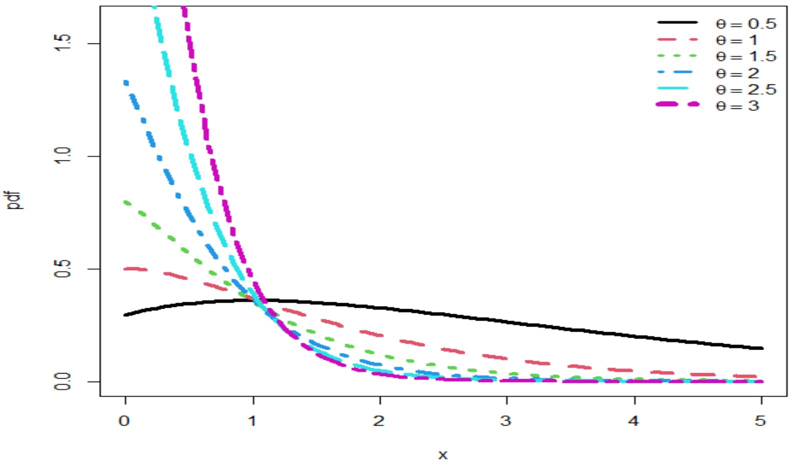

Figure (1): The (pdf) of the Lindley Function with different values of parameters θ.

The cumulative density function (CDF) of one parameter Lindley distribution, corresponding to the pdf given in equation

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta}\ e^{-\theta x} \quad (2)$$

Figure (2) shows the (CDF) of the Lindley Function with different parameters θ values.

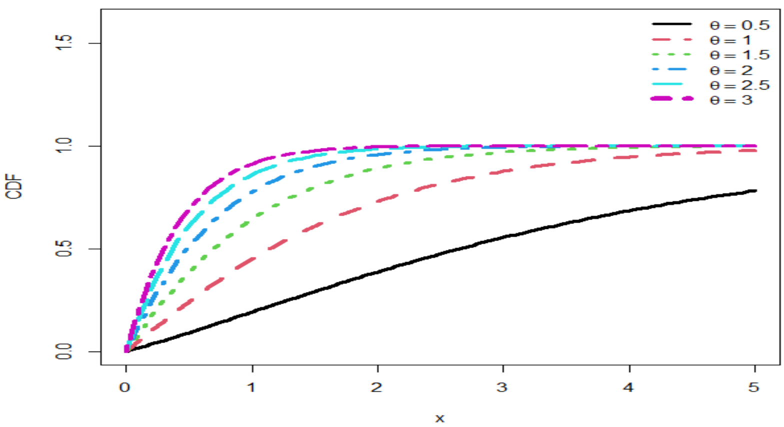

Figure (2): shows the (CDF) of the Lindley Function with different values of parameters θ.

This distribution is derived as a mixture of exponential (θ) and Gamma (2,θ) distribution. Hence the pdf takes the alternate form,

$$G(x) = pf_1(x,\theta) + (1-p)f_2(x,\theta), \quad (3)$$

where $f_1(x,\theta) = \theta e^{-\theta x}$ and $f_2(x,\theta) = \theta^2 x e^{-\theta x}$ are the pdfs of exponential and gamma distributions respectively and $p = \frac{\theta}{1+\theta}$ is the mixing proportion of distributions.

2.1 Reversed Hazard Rate Function:

The Reversed Hazard Rate Function (RHRF) [Ghitany, 2008] is defined as: $RHRF(x,\theta) = \frac{f(x,\theta)}{F(x,\theta)}$,

$$\text{RHRF}(x,\theta) = \frac{\theta^2}{(1+\theta)}(1 + x)\ e^{-\theta x}/(1 - \frac{1+\theta+\theta x}{1+\theta}\ e^{-\theta x}). \qquad (4)$$

Figure (3) shows The value of the Lindley Reversed Hazard Rate Function with different parameters θ values.

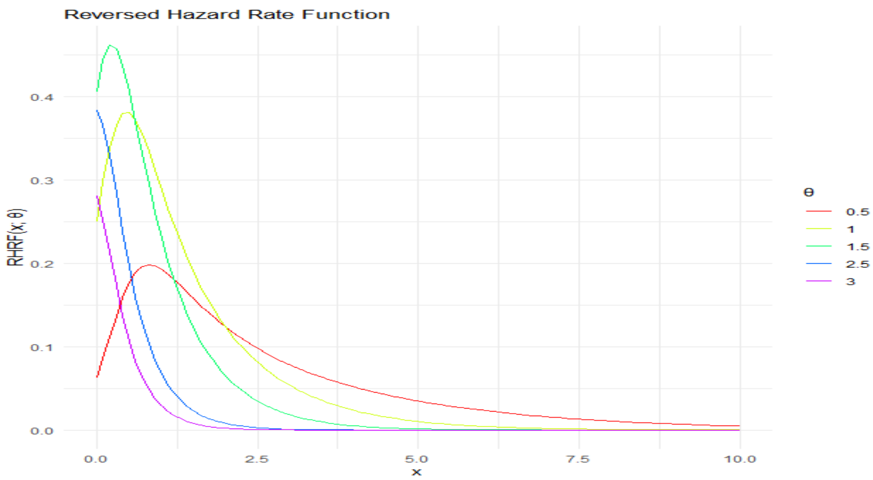

Figure (3): The value of the Lindley Reversed Hazard Rate Function with parameters θ.

The survival analysis, can be represented as the complement of its cumulative distribution function[Belhamra *et al*,(2022), Safari *et al*,(2020)]. Figure 4 depicts the reliability function for various theta values, given by:

$$R(x,\theta) = 1 - (1 - \frac{1+\theta+\theta x}{1+\theta}\ e^{-\theta x}) \qquad (5)$$

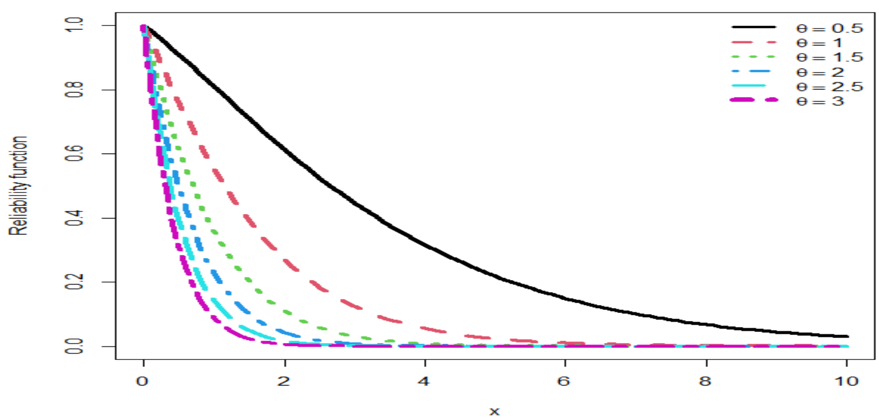

Figure (4): the survival analysis (Reliability Function) of the Lindley Distribution.

The Hazard Rate Function can be calculated as [Belhamra *et al*,(2022)]:

$$h(x,\theta) = \frac{f(x)}{1-F(x)} = \frac{\frac{\theta^2}{(1+\theta)}\ (1+x)\ e^{-\theta x}}{1-(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})} \qquad (6)$$

Figure (5) shows The value of the Lindley Reversed Hazard Rate Function with different parameters θ values

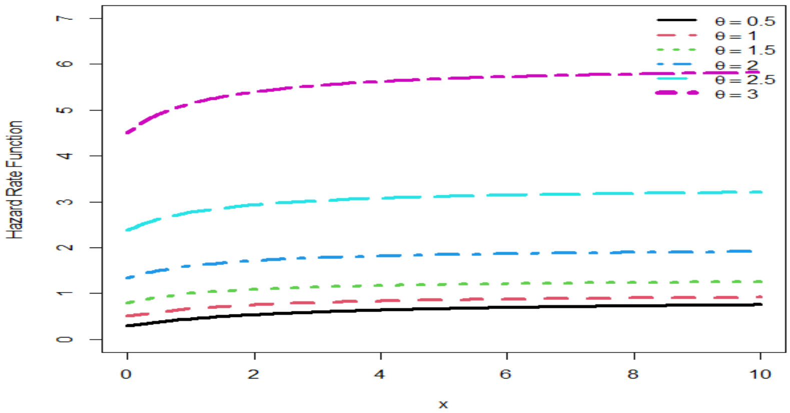

Figure (5): Hazard Rate Function

2.2 Mean Residual Life:

The Mean Residual Life can be calculated as[Irshad, (2017)]:

$$MRL(x,\theta) = \frac{\theta^2 \ x(1+x) \ e^{-\theta x}}{(1+\theta)^2 \ (1-e^{-\theta t})} + t. \quad (7)$$

Figure (6) shows the value of the Lindley Mean Residual Life with different parameters θ values

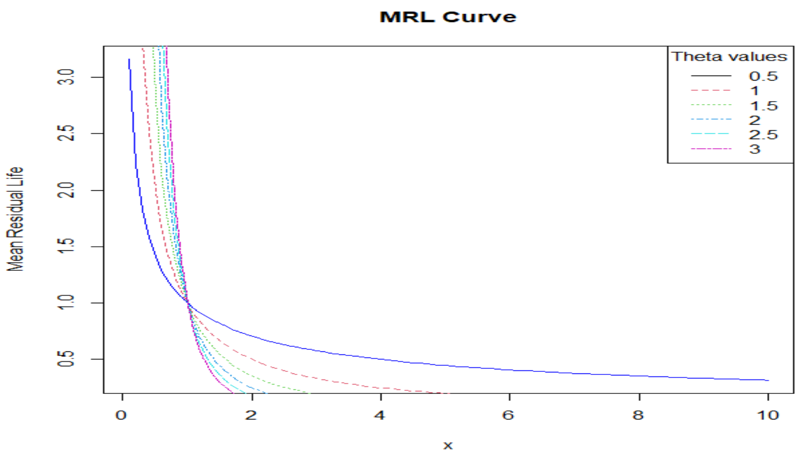

Figure (6): Mean Residual Life

2.3 Mean Inactivity Time:

Mean Inactivity Time= $\int_0^\infty 1 - F(x+t)dt,$

$$\text{MIT}= \int_0^\infty \frac{1-e^{-\theta(x+t)}-\theta}{(1+\theta)(x+t)e^{-\theta(x+t)}} dt. \quad (8)$$

Figure (7) shows The value of the Lindley Mean Inactivity Time with parameters $\theta = 0.5 , 1, 1.5, 2, 2.5, 3.$

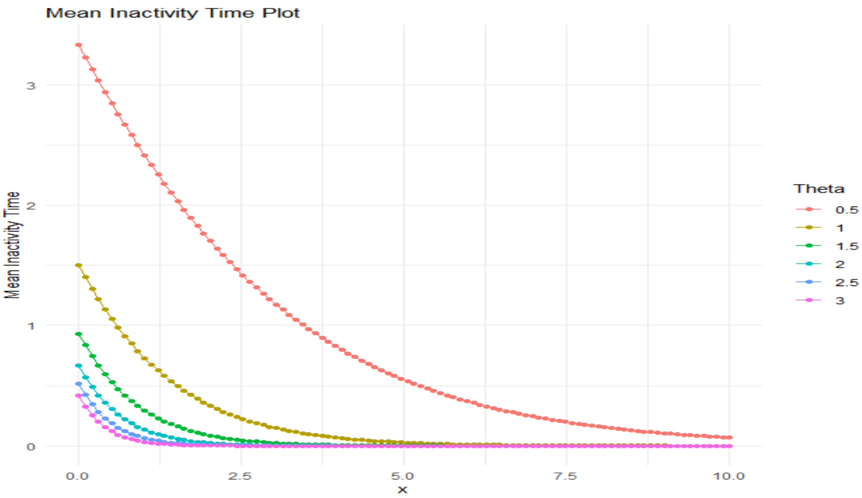

Figure (7): Mean Inactivity Time

2.4 Stress-Strength Reliability:

The Stress-Strength Reliability (SSR) is calculated as follows [Abdi et al., 2019]: Let us consider X and Y as two distinct Lindley Random Variables (RVs) with individual parameters denoted as θ_1 and θ_2. These RVs have Probability Density Functions represented as $f_X(\cdot)$ and $f_Y(\cdot)$. It is crucial to emphasize that X and Y are independent of each other:

$$SSR = P(Y < X) = \int_{stress}^{\infty} \mathrm{p}(\mathrm{Y} < \mathrm{X}|\mathrm{Y} = \mathrm{y}) f_Y(\mathrm{y}) \mathrm{dy} \quad (9)$$

The integral is performed over the range from stress to infinity, integrating the joint probability of stress exceeding strength for all possible strength values. Figure (8) shows the value of the Lindley Stress-Strength Reliability with different parameters θ values.

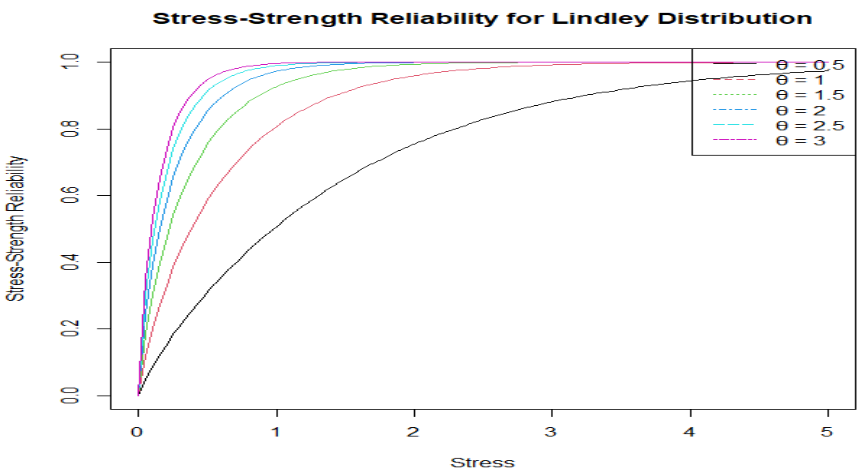

Figure (8): Stress-Strength Reliability with parameters θ.

3. Various Estimation Techniques

3.1 Maximum Likelihood Estimator:

This section describes the process of obtaining Maximum Likelihood Estimates (MLEs) for the unknown parameters of a Lindley distribution (θ). Let's consider a sample of size n from the Lindley distribution as $x = (x_1, x_2, \dots, x_n)$. The likelihood function, based on the observed data, can be expressed as follows[Ghitan,2008]:

$$L(x_1, x_2, \dots, x_n|\theta) = \frac{\theta^{2n}}{(1+\theta)^n} \prod_{i=1}^{n} (1 + x_i) e^{\theta \sum_{i=1}^{n} x_i}, \quad (10)$$

by taking the logarithm for equation (10) we have

$$lnL(\theta, x_1, x_2, \dots, x_n) = 2nln\theta - nln(1+\theta) + ln\prod_{i=1}^{n}(1+x_i) - \theta\sum_{i=1}^{n} x_i, \qquad (11)$$

partially differentiation equation (11) we have

$$\frac{\partial lnL(\theta, x_1, x_2, \dots, x_n)}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{(1+\theta)} - \sum_{i=1}^{n} x_i = \frac{2n(\theta+1) - n\theta - \theta(\theta+1)\sum_{i=1}^{n} x_i}{\theta(\theta+1)}, \qquad (12)$$

hence equation (12) is equal to 0, then we obtain

$$2(\hat{\theta}+1) - \hat{\theta} - \hat{\theta}(\hat{\theta}+1)\bar{x} = 0,$$

⋮

$$\bar{x}\hat{\theta}^2 + (\bar{x}-1)\hat{\theta} - 2 = 0.$$

And

$$\hat{\theta}_{ML} = \frac{-(1-\bar{x}) + \sqrt{(\bar{x}-1)^2 + 8\bar{x}}}{2\bar{x}}, \bar{x} > 0 \qquad (13)$$

3.2 Least squares method

Consider a random sample e $X_1, X_2, \cdots, X_n$ drawn from the Lindley Distribution and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ represent the corresponding ordered sample. In the context of estimating the parameters of the Lindley Distribution, the Least Squares (LS) method is employed. This method aims to minimize the sum of squared differences between the observed and predicted values. To achieve this, the derivative of the sum of squared differences with respect to the parameter is computed and equated to zero. The sum of squared differences is mathematically defined as follows[Hassan,2019]:

$$l(x,\theta) = \sum_{i=1}^{n}(F(x_{(i)}) - \frac{i}{n+1})^2 . \qquad (14)$$

To find the value of θ that minimizes the sum of squared differences we can formulate the equations as follows:

$$\hat{\theta} = \text{argmin}\theta \sum_{i=1}^{n}\left[F(x_{(i)},\theta) - \frac{i}{(n+1)}\right]^2. \qquad (15)$$

Where

- $\hat{\theta}$ is the estimated value of θ.
- $F(X_{(i)},\theta)$ is the distribution function with parameter θ.
- $X_{(i)}$ is order sample.
- n is the sample size.

The equations represents an optimization problem where we are searching for the value of θ that minimizes the sum of squared differences between the observed quantiles $\frac{i}{(n+1)}$ and the quantiles predicted by the distribution function $F(x,\theta)$. The argmin operation denotes the value of θ that achieves the minimum of the objective function.

3.3 Linear Quantile Moment Method

Obtaining the Quantile Function involves deriving it from the Cumulative Function (F(x)), as outlined as follows

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}.$$

To find the value of x in terms of the CDF and θ, we'll rearrange the equation to isolate x:

$$1 - F(x) = \frac{1 + \theta + \theta x}{1 + \theta} * e^{-\theta x}.$$

Now, multiply both sides by (1 + θ) to get rid of the denominator:

$$(1 + \theta) * (1 - F(x)) = (1 + \theta + \theta x) * e^{-\theta x}.$$

Expand both sides:

$$1 - F(x) + \theta - \theta F(x) = e^{-\theta x} + \theta e e^{-\theta x} + \theta x e^{-\theta x}.$$

Combine like terms:

$$\theta - \theta F(x) = (1 + \theta + \theta x) * e^{-\theta x}.$$

Now, isolate x on one side:

$$\theta - \theta F(x) = \theta e^{-\theta x} + (1 + \theta)e^{-\theta x}.$$

Factor out $e^{-\theta x}$

$$\theta - \theta F(x) = (\theta + 1) * e^{-\theta x} + \theta x e^{-\theta x}.$$

Subtract (θ + 1) * $e^{-\theta x}$ from both sides:

$$\theta - \theta F(x) - (\theta + 1) * e^{-\theta x} = \theta x e^{-\theta x}.$$

Divide both sides by θ: $1 - F(x) - \frac{\theta + 1}{\theta} * e^{-\theta x} = x * e^{-\theta x}$,

now, $x = \frac{1 - F(x) - \frac{\theta + 1}{\theta} * e^{-\theta x}}{e^{-\theta x}}$.

The equation mentioned above represents the Quantile function (Quantile function), which can be denoted in the following manner:

$$Q(F) = \frac{1 - F(x) - \frac{\theta + 1}{\theta} * e^{-\theta x}}{e^{-\theta x}},$$

the Quantile moments of a random sample of size n: $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. As follows

$$\hat{\epsilon}_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r}{k} \hat{\tau}_{p,m}(X_{r-k:r}). \quad (16)$$

Where $X_{r-k:r}$ represents the ordered sample values from X_{r-k} to X_r. $\hat{\tau}_{p,m}(X_{r-k:r})$ is the quantile estimator defined as a weighted combination of quantile estimates:

$$\hat{\tau}_{p,m}(X_{r-k:r}) = p\hat{Q}_{r-k:r}(m) + (1 - 2p)\hat{Q}_{r-k:r}\left(\frac{1}{2}\right) + p\hat{Q}_{r-k:r}(1 - m), \quad (17)$$

p is the quantile level typically set to 0.5 for median estimation, m is a parameter that determines the fraction of the quantile level on each side typically set to 0.5 for symmetric estimation. And $\hat{Q}_{r-k:r}(u)$ is the quantile estimator obtained using the sample data.

$$\hat{Q}_{r-k:r}(u) = \sum_{i=1}^{n}\left[\frac{1}{n} k_h\left(\sum_{j=1}^{i} w_{j,n} - u\right)\right] X_{i,n}, \quad (18)$$

where $k_{h(t)}$ is the kernel function which is defined as the standard normal density function:

$$k_h(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad (19)$$

and $w_{j,n}$ are the weights used in the quantile estimator defined as

$$w_{j,n} = \begin{cases} \frac{1}{2}\left(1 - \frac{n-2}{\sqrt{n(n-1)}}\right), & if\ i = 1, n \\ \frac{1}{\sqrt{n(n-1)}}, & if\ i = 1,2, \ldots, n-1 \end{cases} \quad (20)$$

with the aim of deriving estimations through the LQM approach, equation 16 is employed, and the R programming language is utilized to determine the parameter.

4. Application

4.1 Simulation study:

A simulation study will be conducted to compare the performance of three different estimators: the maximum likelihood estimators (MLEs), LQM estimators, and the least squares method estimators. The comparison will primarily focus on the estimates and mean squared errors (MSEs). The study will consider various sample sizes (25, 50, 75, 100, 150), utilize the R program, and explore different values for the theta parameters. To cover all possible combinations of sample size and shape parameter values, the experiment will be repeated 1000 times. The results, including the estimated parameters and MSEs for and, will be presented in Tables 1, 2, 3, 4, 5, and 6.

Table 1: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25, 50, 100, 150) For the initial value set.

methods	Sample size	value	estimate	Statistics		
		θ	$\hat{\theta}$	MSE($\hat{\theta}$)	AIC	BIC
MLE	25	0.5	0.8705713	0.1373231	-7.94376	-3.50601
LQM			0.3866515	0.4393473	80.90132	85.33907
LS			-0.049325	0.3017577	75.00254	76.22142
MLE	50	0.5	0.7021932	0.04088208	1.677466	7.501512
LQM			1.24793	0.5593987	184.1752	189.9992
LS			-0.026124	0.2768065	130.1272	132.0392
MLE	100	0.5	0.751179	0.06309089	-15.8903	-8.68004
LQM			1.187028	0.4720077	349.7226	56.9329
LS			-0.032666	0.2837332	280.0331	282.6383
MLE	150	0.5	0.7810404	0.07898371	-27.6034	-19.5822
LQM			1.162442	0.4388296	509.4133	517.4346
LS			-0.037883	0.289319	403.3566	406.3672

In Table 1, the results indicate that the Maximum Likelihood Estimation (MLE) method consistently exhibits superior performance compared to the Least Squares (LS) and (LQM) methods in terms of Mean Squared Error (MSE). Across the various sample sizes and the initial value set, the MLE method consistently achieves smaller MSE values, suggesting that it provides more accurate parameter estimates. These findings support the notion that the MLE method is highly effective in capturing the underlying parameter values, thus making it a favorable choice for parameter estimation in this context.

Table 2: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25,50,100,150) For the initial value set.

methods	Sample size	value	estimate	Statistics		
		θ	$\hat{\theta}$	MSE($\hat{\theta}$)	AIC	BIC
MLE	25	1	1.575181	0.3308332	45.85717	50.29492
LQM			0.9603409	0.001572844	-35.7644	-31.3267
LS			-0.233721	1.52207	90.1945	91.41338
MLE	50	1	1.273744	0.07493584	114.4008	120.2248
LQM			1.075886	0.005758614	-42.4197	-36.5957
LS			-0.135957	1.290398	162.4002	164.3123
MLE	100	1	1.361266	0.1305133	210.3338	217.5441
LQM			1.006858	4.70332e-05	-111.836	-104.626
LS			-0.168178	1.364642	339.4725	342.0777
MLE	150	1	1.414686	0.1719645	300.6415	308.6628
LQM			0.9772737	0.000516483	-178.255	-170.233
LS			-0.190060	1.416244	498.7892	501.7998

Tables 2 indicate that, considering the MSE values and the accuracy of parameter estimation, the LQM method appears to be the most favorable option among the three methods.

Table 3: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25,50,100,150) For the initial value set.

methods	Sample size	value	estimate	Statistics		
		θ	$\hat{\theta}$	MSE($\hat{\theta}$)	AIC	BIC
MLE	25	1.5	2.228961	0.5313839	25.48987	29.92762
LQM			0.8031283	0.4856302	-59.6206	-55.18294
LS			-0.525521	4.102736	98.09339	99.31227
MLE	50	1.5	1.800792	0.0904758	73.77994	79.60399
LQM			0.9332349	0.3212227	-82.2565	-76.4325
LS			-0.317829	3.304503	179.6108	181.5228
MLE	100	1.5	1.924846	0.1804942	129.2684	136.4787
LQM			0.8610775	0.4082219	-197.285	-190.0754
LS			-0.392166	3.580293	371.181	373.7862
MLE	150	1.5	2.000671	0.2506714	179.1448	187.166
LQM			0.8294724	0.4496072	-311.190	-303.1695
LS			-0.439193	3.760473	548.6645	551.6751

Table 4: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25,50,100,150) For the initial value set.

methods	Sample size	value	estimate	Statistics		
		θ	$\hat{\theta}$	MSE($\hat{\theta}$)	AIC	BIC
MLE	25	2	2.857448	0.7352175	11.07124	15.50899
LQM			0.6817638	1.737747	-80.1915	-75.75379
LS			-0.899050	8.404492	103.2536	104.4725
MLE	50	2	2.304725	0.0928573	44.9961	50.82014
LQM			0.8159113	1.402066	-117.750	-111.9264
LS			-0.555618	6.531186	190.9856	192.8977
MLE	100	2	2.464591	0.2158446	71.83609	79.04643
LQM			0.7440355	1.577447	-272.657	-265.4476
LS			-0.683930	7.203482	392.1646	394.7698
MLE	150	2	2.562419	0.316315	93.04789	101.0692
LQM			0.7123558	1.658027	-427.700	-419.6791
LS			-0.762269	7.630135	581.2866	584.2973

Table 5: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25,50,100,150) For the initial value set.

methods	Sample size	value θ	estimate $\hat{\theta}$	Statistics MSE($\hat{\theta}$)	AIC	BIC
MLE	25	2.5	3.470914	0.9426742	-0.10178	4.335973
LQM			0.587565	3.657408	-98.0881	-93.65042
LS			-1.33747	14.72617	107.0101	108.229
MLE	50	2.5	2.794521	0.0867424	22.67783	28.50187
LQM			0.7195976	3.169833	-149.459	-143.6355
LS			-0.838620	11.14639	199.3129	201.225
MLE	100	2.5	2.98989	0.2399921	27.30121	34.51155
LQM			0.6500397	3.422353	-339.365	-332.1551
LS			-1.029808	12.45954	407.5564	410.1616
MLE	150	2.5	3.109555	0.3715573	26.27797	34.29924
LQM			0.6193887	3.536699	-530.26	-522.2388
LS			-1.144101	13.27947	605.0438	608.0544

Table 6: MSE of the parameter estimations and a comparison of the three methods of estimation at the sample sizes (25,50,100,150) For the initial value set.

methods	Sample size	value θ	estimate $\hat{\theta}$	Statistics MSE($\hat{\theta}$)	AIC	BIC
MLE	25	3	4.074537	1.154631	-9.22625	-4.788501
LQM			0.513574	6.182314	-113.804	-109.3667
LS			-1.829306	23.3222	109.9269	111.1457
MLE	50	3	3.274838	0.0755357	4.443963	10.26801
LQM			0.6402976	5.568195	-177.931	-172.1077
LS			-1.159422	17.30079	205.7955	207.7076
MLE	100	3	3.505572	0.255603	-9.08911	-1.878777
LQM			0.5741024	5.884979	-398.757	-391.5473
LS			-1.420498	19.5408	419.5642	422.1694
MLE	150	3	3.647004	0.4186145	-28.2847	-20.26344
LQM			0.5450135	6.026959	-621.154	-613.1331
LS			-1.574407	20.9252	623.4893	626.4999

Tables 3 through 6 indicate that considering the mean squared error (MSE) values and the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) statistics, the Maximum Likelihood Estimators (MLE) method seems to be the most appropriate option among the three methods, particularly when dealing with larger sample sizes.

4.2 Real data

This research applied theoretical principles to analyze the time spent by customers who hold current accounts with Zanko Bank/University of Duhok. The sample included 55 customers, and their time durations were recorded while completing the various steps involved in the bank withdrawal process. These steps include providing information about the withdrawal process and subsequently receiving the amounts of the checks from the treasurer. Table 7 displays the waiting times of bank customers.

Table 7: presents the duration of time that customers spend at the bank, measured in hours

0.91666	0.2833	0.5833	0.25	0.2	0.4833	0.35	0.3833
0.45	0.6833	0.81666	0.8166	0.25	0.4333	0.2833	0.4166
0.7	0.1833	0.18333	0.31666	0.3666	0.3666	0.3500	0.8833
0.6666	0.6666	1.26666	0.850	0.6333	0.5333	0.5833	0.3500
0.4833	0.1166	0.53333	0.4000	0.3166	0.7833	0.2833	0.51666
0.5500	0.1666	0.3000	0.28333	0.350	0.2166	0.3500	0.5333
0.533	0.5000	0.2333	0.6000	0.1333	0.2333	0.1333	

Table 8: Presents the results of the Goodness-of-Fit tests conducted on the waiting time data.

methods	estimate	Statistics				
	$\hat{\theta}$	MSE($\hat{\theta}$)	AIC	BIC	Kolmogorov Smirnov	p-value
MLE	2.77693	0.0219635	22.66204	28.67671	0.19477	0.07084
LQM	0.4739658	0.1801348	-171.742	-165.7279		
LS	-1.616467	52.8247	145.6111	147.6184		

The test statistic (D) for the Kolmogorov-Smirnov test was 0.19477, with a p-value of 0.07084. The test's null hypothesis is that the data follows the Lindley distribution. We do not have enough evidence to reject the null hypothesis because the p-value (0.07084) is greater than the commonly used significance level of 0.05. As a result of this test.

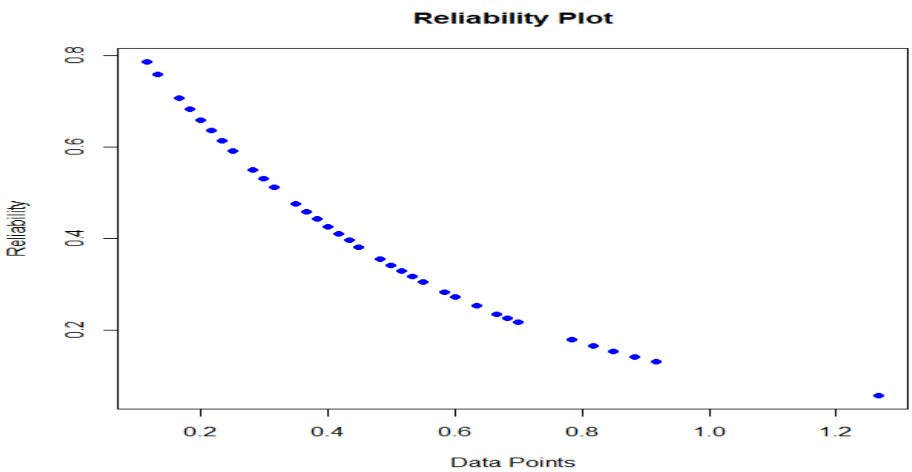

Figure (9): Reliability of time that customers spend at the bank, measured in hours

5. Conclusion

In conclusion, based on the simulation study and the analysis of real data, the Maximum Likelihood Estimation (MLE) method demonstrated superior performance in parameter estimation, particularly for larger sample sizes. Therefore, it can be considered the most appropriate and accurate method for capturing the underlying parameter values in this context. Researchers and practitioners can confidently use the MLE method to estimate parameters in similar situations. However, further investigations on different datasets and scenarios may be beneficial to validate the robustness and generalizability of these findings.

Availability of Data: The datasets that support the paper's results are included in the paper.

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