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Schur Convexity of Complementary Geometric Mean

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Abstract:*In this paper, the generalized forms of complementary geometric mean are introduced. Further, studied the various properties like homogeneous, isotone and convexity.Also, discussed theoscillatory mean involving complementary geometric and arithmetic means, various types of Schur convexities results.*

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Keywords: *Arithmetic mean, Geometric mean, Harmonic mean, Contra harmonic mean, G-Complementary mean, Convexity.*

1. Introduction

The arithmetic mean $A(Pi\,,Pj)\ =\frac{P_i+P_j}{2}$, Geometric mean $G(Pi\,,Pj)=\sqrt{P_iP_j}$, Harmonic mean $H(Pi\,,Pj)=\frac{2P_iP_j}{P_i+P_j}$ and Contra harmonic mean $C(Pi\,,Pj)=\frac{P_i^2+P_j^2}{P_i+P_j}$have their own importance in literature.

Convexity results on means refer [14, 20] and Schur convexity results were found in [1, 2, 6, 15-17, 21, 23] of one function with respect to another is investigated. The convexity results on some standard means with their applications to mean inequalities and applications to few Greek means in Engineering field were also discussed in [5, 7-9, 11-13, 22, 25].

Somedistinct and interesting results on generalization of means and properties of well-known means were found in[3, 4, 10,18, 19, 24].

2. Definitions and Lemmas

In this section, recall some definitions and lemmas necessary to develop this paper.

Definition 2.1: If $0\ <P_i<P_j\ ,2A-G$ is called complementary geometric mean with respect to A or complementary geometric mean is given by${}^CG = P_i\ -\sqrt{P_iP_j}+P_j$.

Generalised complementary geometric mean is given by

$${}^CG_n = \frac{2}{n}\sum_{i=1}^{n} P_i\ -(\prod_{i=1}^{n} P_i\,)^{1/n}$$

Generalised complementary weighted geometric mean is given by

$^{C}GW_n = \frac{2}{n}\sum_{i=1}^{n} W_i\, P_i\ - (\ \prod_{i=1}^{n} W_i\, P_i\ )^{1/n}$

Definition 2.2: For $0 < P_i < P_j$ and $\mu \in (0, 1)$, the oscillatory mean of complementary geometric mean and arithmetic mean is given by

$O_{CGA}(P_i, P_j\ ;\ \mu) = \mu\ (P_i\ - \sqrt{P_i\, P_j} + P_j) + (1-\mu)\frac{P_i\ + P_j}{2}$ and its dual is given by

$O_{CGA}^{(d)}(P_i, P_j\ ;\ \mu) = (P_i\ - \sqrt{P_i\, P_j} + P_j\ )^{\mu}\ (\frac{P_i + P_j}{2})^{1-\mu}$

Definition 2.3: For $0 < P_i < P_j$ and $\mu \in (0, 1)$, the oscillatory mean of complementary geometric mean and geometric mean is given by

$O_{CGG}\ (P_i, P_j\ ;\ \mu) = \mu\ (P_i\ - \sqrt{P_i\, P_j} + P_j) + (1-\mu)\sqrt{P_i\, P_j}$ and its dual is given by

$O_{CGG}^{(d)}(P_i, P_j\ ;\ \mu) = (P_i\ - \sqrt{P_i\, P_j} + P_j\ )^{\mu}\ (P_i\, P_j)^{\frac{(1-\mu)}{2}}$

2.1 Remark: For $0 < P_i < P_j$, then

i. $P_j - {}^{C}G = P_j - (P_i\ - \sqrt{P_i\, P_j} + P_j) = G - P_i\ \geq 0$

ii. ${}^{C}G - P_i\ = (P_i\ - \sqrt{P_i\, P_j} + P_j) - P_i\ = P_j - G\ \geq 0$

It is a clear indication that $P_i\ \leq {}^{C}G \leq P_j$ and it is justified that ${}^{C}G$ is a mean.

2.2 Properties: For any two positive real numbers P_i and P_j , then

i. ${}^{C}G\ (P_i, P_i\ ) = P_i\ -\ P_i\ +\ P_i\ = P_i$

ii. ${}^{C}G(P_j, P_i\ ) = (P_j\ - \sqrt{P_i\, P_j} + P_i\ ) = P_i\ -\ \sqrt{P_i\, P_j}\ +\ P_j = {}^{C}G(P_i, P_j\ )$

iii. ${}^{C}G\ (\ tP_i, tP_j\ ) = tP_i - \sqrt{(tP_i\ )(tP_j)}\ +\ tP_j = t\ (P_i\ - \sqrt{P_i\, P_j} + P_j\ ) = t\ {}^{C}G(P_i, P_j\ )$

iv. ${}^{C}G(x, P_j\ ) - {}^{C}G\ (P_i, x) = (x - \sqrt{xP_j} + P_j\ ) - (\ P_i - \sqrt{(\ P_i\, x)} + x)$

$$= (\sqrt{P_j} - \sqrt{P_i}\ )\ (\sqrt{P_j} + \sqrt{P_i} - \sqrt{x}\ ) > 0$$

Therefore, ${}^{C}G(P_i, P_j\ )$ is Reflexive, Symmetric, Homogeneous and Isotone.

2.3 Inequality: For $0 < P_i < P_j$, $A < {}^{C}G < C$ holds.

${}^{C}G - A = (P_i\ -\ \sqrt{P_i\, P_j}\ +\ P_j) - \left(\frac{P_i + P_j}{2}\right) = \left(\frac{P_i + P_j}{2}\right) - \sqrt{P_i\, P_j} > 0$

$C - {}^{C}G = \frac{P_i^2 + P_j^2}{P_i + P_j} - (P_i\ - \sqrt{P_i\, P_j} + P_j) = \frac{\sqrt{P_i\, P_j}}{P_i + P_j}\ (P_i\ - 2\ \sqrt{P_i\, P_j} + P_j) > 0$

Therefore, A $<^{C}G < C$.

Lemma 2.1: A mean M is called convex (with respect to A) if

$$(2.1)\qquad \begin{vmatrix} 1 & A(P_i) & M(P_i) \\ 0 & A(P_j) - A(P_i) & M(P_j) - M(P_i) \\ 0 & A(P_k) - A(P_i) & M(P_k) - M(P_i) \end{vmatrix} \geq 0$$

Lemma 2.2: Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set and let $\varphi : \Omega \rightarrow R_+$ be continuous on Ω and differentiable in Ω^0 .

If φ is symmetric on Ω and

$$(2.2)\qquad (P_i - P_j)\left[\frac{\partial\varphi}{\partial P_i} - \frac{\partial\varphi}{\partial P_j}\right] \geq 0\ (\leq 0)$$

$$(2.3)\qquad (\ln P_i - \ln P_j)\left[P_i \frac{\partial\varphi}{\partial P_i} - P_j \frac{\partial\varphi}{\partial P_j}\right] \geq 0\ (\leq 0)$$

$$(2.4)\qquad (P_i - P_j)\left[P_i^{\,2} \frac{\partial\varphi}{\partial P_i} - P_j^{\,2} \frac{\partial\varphi}{\partial P_j}\right] \geq 0\ (\leq 0)$$

holds for any $P = (P_1, P_2, \ldots, P_n) \in \Omega^0$, $0 < P_i < P_j$ then ϕ is a Schur convex (concave), Schur geometrically convex (concave) and Schur harmonically convex (concave) function respectively.

3. Main Results

Theorem 3.1: The complementary geometric mean is convex with respect to arithmetic mean.

Proof: For $0 < P_i < P_j < P_k$, then by lemma 2.1,

$$(3.1)\qquad \begin{vmatrix} 1 & A(P_i) & C_G(P_i) \\ 0 & A(P_j) - A(P_i) & c_G(P_j) - c_G(P_i) \\ 0 & A(P_k) - A(P_i) & c_G(P_k) - c_G(P_i) \end{vmatrix}$$

Apply the definitions of arithmetic and complementary geometric mean to (3.1) gives

$$(3.2)\qquad = \begin{vmatrix} 1 & \frac{P_i + 1}{2} & P_i - \sqrt{P_i} + 1 \\ 0 & \frac{P_j - P_i}{2} & \left(P_j - \sqrt{P_j}\right) - \left(P_i - \sqrt{P_i}\right) \\ 0 & \frac{P_k - P_i}{2} & \left(P_k - \sqrt{P_k}\right) - \left(P_i - \sqrt{P_i}\right) \end{vmatrix}$$

Further simplification leads to,

$$(3.3)\qquad = \frac{\left(\sqrt{P_j} - \sqrt{P_j}\right)\left(\sqrt{P_k} - \sqrt{P_j}\right)}{2} \begin{vmatrix} 1 & P_i + 1 & P_i - \sqrt{P_i} + 1 \\ 0 & \sqrt{P_j} + \sqrt{P_i} & \sqrt{P_j} + \sqrt{P_i} - 1 \\ 0 & \sqrt{P_k} + \sqrt{P_i} & \sqrt{P_k} + \sqrt{P_i} - 1 \end{vmatrix}$$

$$= \frac{(\sqrt{P_j} - \sqrt{P_i})(\sqrt{P_k} - \sqrt{P_i})}{2(\sqrt{P_k} - \sqrt{P_j})} \geq 0$$

Therefore, complementary geometric mean is convexwith respect to arithmetic mean.

Theorem 3.2: The oscillatory mean of complementary geometric mean and arithmetic mean, denoted by $O_{CGA}(P_i, P_j; \mu)$, $\mu \in (0, 1), 0 < P_i < P_j$ and its dual are Schur convex.

Proof: Case 1

Consider, $O_{CGA}(P_i, P_j; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1 - \mu)\left(\frac{P_i + P_j}{2}\right)$

$$\frac{\partial}{\partial P_i}[OCGA(P_i, P_j; \mu)] = \frac{\partial}{\partial P_i}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1 - \mu)\left(\frac{P_i + P_j}{2}\right)\right]$$

$$= \frac{\mu}{2} + \frac{1}{2} - \frac{\mu P_j}{2\sqrt{P_i P_j}}$$

$$\frac{\partial}{\partial P_j}[O_{CGA}(P_i, P_j; \mu)] = \frac{\partial}{\partial P_j}\left[\mu\left(P_i - \sqrt{P_i P_j} + P_j\right) + (1 - \mu)\left(\frac{P_i + P_j}{2}\right)\right]$$

$$= \frac{\mu}{2} + \frac{1}{2} - \frac{\mu P_i}{2\sqrt{P_i P_j}}$$

$$\frac{\partial}{\partial P_i}[OCGA(P_i, P_j; \mu)] - \frac{\partial}{\partial P_j}[O_{CGA}(P_i, P_j; \mu)] = \frac{\mu(P_i - P_j)}{2\sqrt{P_i P_j}}$$

Therefore, for $0 < P_i < P_j$,

$$(P_i - P_j)\left\{\frac{\partial}{\partial P_i}[OCGA(P_i, P_j; \mu)] - \frac{\partial}{\partial P_j}[OCGA(P_i, P_j; \mu)]\right\} \geq 0$$

Case 2:

Consider, $O^{(d)}_{CGA}(P_i, P_j; \mu) = (P_i - \sqrt{P_i P_j} + P_j)^{\mu} \left(\frac{P_i + P_j}{2}\right)^{1-\mu}$

$$ln\ O^{(d)}_{CGA}(P_i, P_j; \mu) = \mu ln\ (P_i - \sqrt{P_i P_j} + P_j) + (1 - \mu)\ \ln\left(\frac{P_i + P_j}{2}\right)$$

$$\frac{\partial}{\partial P_i}[O^{(d)}_{CGA}(P_i, P_j; \mu)] = O^{(d)}_{CGA}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$$

$$\frac{\partial}{\partial P_j}[O^{(d)}_{CGA}(P_i, P_j; \mu)] = O^{(d)}_{CGA}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$$

$\frac{\partial}{\partial P_i}[O^{(d)}_{CGA}(P_i, P_j ; \mu)] - \frac{\partial}{\partial P_j}[O^{(d)}_{CGA}(P_i, P_j ; \mu)] = O^{(d)}_{CGA}(P_i, P_j ; \mu)\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}[\frac{(P_i - P_j)}{2\sqrt{P_i P_j}}]$

Therefore, for $0 < P_i < P_j$,

$(P_i - P_j)\left\{\frac{\partial}{\partial P_i}[O^{(d)}_{CGA}(P_i, P_j ; \mu)] - \frac{\partial}{\partial P_j}[O^{(d)}_{CGA}(P_i, P_j ; \mu)]\right\} \geq 0$

By case 1 and case 2, $O_{CGA}(P_i, P_j ; \mu)$ and its dual are Schur Convex.

Theorem 3.3: The oscillatory mean of complementary geometric mean and arithmetic mean, denoted by $O_{CGA}(P_i, P_j ; \mu)$, $\mu \in (0, 1), 0 < P_i < P_j$ and its dual are Schurgeometric convex for $0 < P_i < P_j$.

Proof: Case 1

Consider, $OCGA(P_i, P_j ; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)$

$P_i\frac{\partial}{\partial P_i}[OCGA(P_i, P_j ; \mu)] = P_i\frac{\partial}{\partial P_i}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)\right]$

$= \frac{\mu P_i}{2} + \frac{P_i}{2} - \frac{\mu P_i P_j}{2\sqrt{P_i P_j}}$

$P_j\frac{\partial}{\partial P_j}[OCGA(P_i, P_j ; \mu)] = P_j\frac{\partial}{\partial P_j}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)\right]$

$= \frac{\mu P_j}{2} + \frac{P_j}{2} - \frac{\mu P_i P_j}{2\sqrt{P_i P_j}}$

$P_i\frac{\partial}{\partial P_i}[OCGA(P_i, P_j ; \mu)] - P_j\frac{\partial}{\partial P_j}[OCGA(P_i, P_j ; \mu)] = (P_i - P_j)\left(\frac{\mu+1}{2}\right)$

Therefore, for $0 < P_i < P_j$,

$(lnP_i - lnP_j)\left\{P_i\frac{\partial}{\partial P_i}[OCGA(P_i, P_j ; \mu)] - P_j\frac{\partial}{\partial P_j}[OCGA(P_i, P_j ; \mu)]\right\} \geq 0$

Case 2:

Consider, $O^{(d)}_{CGA}(P_i, P_j ; \mu) = (P_i - \sqrt{P_i P_j} + P_j)^{\mu}\left(\frac{P_i + P_j}{2}\right)^{1-\mu}$

$ln\, O^{(d)}_{CGA}(P_i, P_j ; \mu) = \mu ln(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\, ln\left(\frac{P_i + P_j}{2}\right)$

$P_i\frac{\partial}{\partial P_i}[O^{(d)}_{CGA}(P_i, P_j ; \mu)] = P_i\, O^{(d)}_{CGA}(P_i, P_j ; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$

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$P_j \frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j ; \mu)] = P_j O_{CGA}^{(d)}(P_i, P_j ; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$

$$P_i \frac{\partial}{\partial P_i}[O_{CGA}^{(d)}(P_i, P_j ; \mu)] - P_j \frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j ; \mu)]$$

$= O_{CGA}^{(d)}(P_i, P_j ; \mu)\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}[P_i - P_j]$

Therefore, for $0 < P_i < P_j$,

$(\ln P_i - \ln P_j)\left\{P_i \frac{\partial}{\partial P_i}[O_{CGA}^{(d)}(P_i, P_j ; \mu)] - P_j \frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j ; \mu)]\right\} \geq 0$

By case 1 and case 2, $O_{CGA}(P_i, P_j ; \mu)$and its dual are Schur geometric convex.

Theorem 3.4: The oscillatory mean of complementary geometric mean andarithmetic mean, denoted by O_{CGA} $(P_i, P_j ; \mu)$, $\mu \in (0, 1)$,$0 < P_i < P_j$ is Schur harmonic convex and its dual is Schur harmonic concave for $0 < P_i < P_j$.

Proof: Case 1

Consider, $O_{CGA}(P_i, P_j ; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)$

$$P_i^2 \frac{\partial}{\partial P_i}[O_{CGA}(P_i, P_j ; \mu)] = P_i^2 \frac{\partial}{\partial P_i}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)\right]$$

$$= P_i^2\left[\frac{\mu}{2} + \frac{1}{2} - \frac{\mu P_j}{2\sqrt{P_i P_j}}\right]$$

$$P_j^2 \frac{\partial}{\partial P_j}[O_{CGA}(P_i, P_j ; \mu)]] = P_j^2 \frac{\partial}{\partial P_j}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\left(\frac{P_i + P_j}{2}\right)\right]$$

$$= P_j^2\left[\frac{\mu}{2} + \frac{1}{2} - \frac{\mu P_i}{2\sqrt{P_i P_j}}\right]$$

$P_i^2 \frac{\partial}{\partial P_i}[O_{CGA}(P_i, P_j ; \mu)] - P_j^2 \frac{\partial}{\partial P_j}[O_{CGA}(P_i, P_j ; \mu)] =$

$(P_i - P_j)\left[\frac{\mu}{2}(P_i - \sqrt{P_i P_j} + P_j) + \frac{P_i + P_j}{2}\right]$

Therefore, for $0 < P_i < P_j$,

$$(P_i - P_j)\left[P_i^{\,2}\frac{\partial}{\partial P_i}\{O_{CGA}(P_i, P_j; \mu)\} - P_j^{\,2}\frac{\partial}{\partial P_j}\{O_{CGA}(P_i, P_j; \mu)\}\right] \geq 0$$

Case 2:

Consider, $O_{CGA}^{(d)}(P_i, P_j; \mu) = (P_i - \sqrt{P_i P_j} + P_j)^{\mu}\left(\frac{P_i+P_j}{2}\right)^{1-\mu}$

$$\ln\ O_{CGA}^{(d)}(P_i, P_j; \mu) = \mu \ln\ (P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\ln\ \left(\frac{P_i+P_j}{2}\right)$$

$$P_i^{\,2}\frac{\partial}{\partial P_i}[O_{CGA}^{(d)}(P_i, P_j; \mu)] = P_i^{\,2}O_{CGA}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$$

$$P_j^{\,2}\frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j; \mu)] = P_j^{\,2}O_{CGA}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i P_j}}\right) + \frac{1-\mu}{P_i + P_j}\right]$$

$$P_i^{\,2}\frac{\partial}{\partial P_i}[O_{CGA}^{(d)}(P_i, P_j; \mu)] - P_j^{\,2}\frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j; \mu)] =$$

$$(P_i^{\,2} - P_j^{\,2})O_{CGA}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu(P_i - P_j)}{(P_i - \sqrt{P_i P_j} + P_j)2\sqrt{P_i P_j}}\right]$$

Therefore, for $0 < P_i < P_j$,

$$(P_i - P_j)\left\{ P_i^{\,2}\frac{\partial}{\partial P_i}[O_{CGA}^{(d)}(P_i, P_j; \mu)] - P_j^{\,2}\frac{\partial}{\partial P_j}[O_{CGA}^{(d)}(P_i, P_j; \mu)]\right\} < 0$$

By case 1 and case 2, $O_{CGA}(P_i, P_j; \mu)$ is Schur harmonic convex and its dual is Schur harmonic concave.

Theorem 3.5: For $0 < P_i < P_j$ and $\mu \in (0, 1)$ Oscillatory mean ofcomplementary geometric mean and geometric mean denoted by $O_{CGG}(P_i, P_j; \mu)$ is Schur convex and its dual is Schur concave.

Proof: Case 1

Consider,

$$O_{CGG}(P_i, P_j; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\sqrt{P_i P_j}$$

$$\frac{\partial}{\partial P_i}O_{CGG}(P_i, P_j; \mu) = \frac{\partial}{\partial P_i}\left[\mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\sqrt{P_i P_j}\right]$$

$$= \mu + (1-2\mu)\frac{P_j}{2\sqrt{P_i P_j}}$$

$$\frac{\partial}{\partial P_j}\, O_{CGG}\,(P_i\,,P_j\,;\mu) = \frac{\partial}{\partial P_j}\left[\mu\,(P_i\ -\sqrt{P_i\,P_j}\ +\ P_j)\ +\ (1\ -\ \mu)\sqrt{P_i\,P_j}\right]$$

$$=\mu + (1-2\mu)\,\frac{P_i}{2\sqrt{P_i\,P_j}}$$

$$\frac{\partial}{\partial P_i}\, O_{CGG}\,(P_i\,,P_j\,;\mu) - \frac{\partial}{\partial P_j}\, O_{CGG}\,(P_i\,,P_j\,;\mu) = (1-2\mu)\,\frac{(P_j - P_i\,)}{2\sqrt{P_i\,P_j}}$$

Therefore, for $0 < P_i < P_j$,

$$(P_i\ - P_j\,)\left\{\frac{\partial}{\partial P_i}\left[\,O_{CGG}\ (P_i\,,P_j\,;\,\mu)\right] - \frac{\partial}{\partial P_j}\,[O_{CGG}\,(P_i\,,P_j\,;\,\mu)]\right\} \geq\ 0$$

Case 2:

Consider,

$$O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu) = (P_i\ -\sqrt{P_i\,P_j} + P_j\,)^{\mu}\,(P_i\ P_j\,)^{\ \frac{(1-\mu)}{2}}$$

$$ln\ \ O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu) = \mu\ ln\ \ (P_i\ -\sqrt{P_i\,P_j} + P_j\ ) + (1-\mu)\ ln\ \ \sqrt{P_i\,P_j}$$

$$\frac{\partial}{\partial P_i}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu) = O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu)\left[\frac{\mu}{P_i\ -\sqrt{P_i\,P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i\,P_j}}\right)\ +\ \frac{(1-\mu)\,P_j}{2\sqrt{P_i\,P_j}}\right]$$

$$\frac{\partial}{\partial P_j}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu) = O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu)\left[\frac{\mu}{P_i\ -\sqrt{P_i\,P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i\,P_j}}\ \right)\ +\ \frac{(1-\mu)\,P_i}{2\sqrt{P_i\,P_j}}\right]$$

$$\frac{\partial}{\partial P_i}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu) - \frac{\partial}{\partial P_j}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu)$$
$$= O_{CGG}^{(d)}\ (P_i\,,P_j\,;\mu)\left[\frac{\mu}{P_i\ -\sqrt{P_i\,P_j} + P_j}\ +\ \mu - 1\right]\frac{(P_i\ - P_j\,)}{2\sqrt{P_i\,P_j}}$$

Therefore, for $0 < P_i < P_j$,

$$(P_i\ - P_j\,)\left[\frac{\partial}{\partial P_i}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\,\mu)\ - \frac{\partial}{\partial P_j}\, O_{CGG}^{(d)}\ (P_i\,,P_j\,;\,\mu)\right] <\ 0$$

By case 1 and case 2, $O_{CGG}\,(P_i\,,P_j\,;\mu)$ is Schur Convex and its dual is Schur Concave.

Theorem 3.6: For $0 < P_i < P_j$ and $\mu \in (0, 1)$, then oscillatory mean of complementary geometric mean and geometric mean and its dual are Schur geometric convex.

Proof: Case 1

Consider, $O_{CGG}(P_i, P_j; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\sqrt{P_i P_j}$

$$\frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) = \mu + (1-2\mu)\frac{P_j}{2\sqrt{P_i P_j}}$$

$$P_i \frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) = \mu P_i + (1-2\mu)\frac{P_j P_i}{2\sqrt{P_i P_j}}$$

$$\frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu) = \mu + (1-2\mu)\frac{P_i}{2\sqrt{P_i P_j}}$$

$$P_j \frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu) = \mu P_j + (1-2\mu)\frac{P_i P_j}{2\sqrt{P_i P_j}}$$

$$P_i \frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) - P_j \frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu) = (P_i - P_j)\mu$$

Therefore, for $0 < P_i < P_j$,

$$(\ln P_i - \ln P_j)\left[P_i \frac{\partial}{\partial P_i} OCGG(P_i, P_j; \mu) - P_j \frac{\partial}{\partial P_j} OCGG(P_i, P_j; \mu)\right] \geq 0$$

Case 2:

Consider, $O_{CGG}^{(d)}(P_i, P_j; \mu) = (P_i - \sqrt{P_i P_j} + P_j)^{\mu}(P_i P_j)^{\frac{(1-\mu)}{2}}$

$$\ln O_{CGG}^{(d)}(P_i, P_j; \mu) = \mu \ln (P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\ln \sqrt{P_i P_j}$$

$$P_i \frac{\partial}{\partial P_i} O_{CGG}^{(d)}(P_i, P_j; \mu) = P_i O_{CGG}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i P_j}}\right) + \frac{(1-\mu)P_j}{2\sqrt{P_i P_j}}\right]]$$

$$P_j \frac{\partial}{\partial P_j} O_{CGG}^{(d)}(P_i, P_j; \mu) = P_j O_{CGG}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i P_j}}\right) + \frac{(1-\mu)P_i}{2\sqrt{P_i P_j}}\right]$$

$$P_i \frac{\partial}{\partial P_i} O_{CGG}^{(d)}(P_i, P_j; \mu) - P_j \frac{\partial}{\partial P_j} O_{CGG}^{(d)}(P_i, P_j; \mu) = O_{CGG}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu(P_i - P_j)}{P_i - \sqrt{P_i P_j} + P_j}\right]$$

Therefore, for $0 < P_i < P_j$,

$$(\ln P_i - \ln P_j)\left[P_i \frac{\partial}{\partial P_i} O_{CGG}^{(d)}(P_i, P_j; \mu) - P_j \frac{\partial}{\partial P_j} O_{CGG}^{(d)}(P_i, P_j; \mu)\right] \geq 0$$

By case 1 and case 2, $O_{CGG}(P_i, P_j; \mu)$ and its dual are Schur geometric Convex.

Theorem 3.7: For $0 < P_i < P_j$ and $\mu \in (0, 1)$,then Oscillatory mean ofcomplementary geometric mean and geometric mean and its dual are Schur harmonic convex.

Proof: Case 1

Consider, $O_{CGG}(P_i, P_j; \mu) = \mu(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\sqrt{P_i P_j}$

$$P_i^2 \frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) = \mu P_i^2 + (1-2\mu)\frac{P_i^2 P_j}{2\sqrt{P_i P_j}}$$

$$P_j^2 \frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu) = \mu P_j^2 + (1-2\mu)\frac{P_j^2 P_i}{2\sqrt{P_i P_j}}$$

$$P_i^2 \frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) - P_j^2 \frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu)$$
$$= \left[\mu(P_i - \sqrt{P_i P_j} + P_j) + \frac{\sqrt{P_i P_j}}{2}\right](P_i - P_j)$$

Therefore, for $0 < P_i < P_j$,

$$(P_i - P_j)\left[P_i^2 \frac{\partial}{\partial P_i} O_{CGG}(P_i, P_j; \mu) - P_j^2 \frac{\partial}{\partial P_j} O_{CGG}(P_i, P_j; \mu)\right] \geq 0$$

Case 2:

Consider, $O_{CGG}^{(d)}(P_i, P_j; \mu) = (P_i - \sqrt{P_i P_j} + P_j)^{\mu}(P_i P_j)^{\frac{(1-\mu)}{2}}$

$$\ln O_{CGG}^{(d)}(P_i, P_j; \mu) = \mu \ln(P_i - \sqrt{P_i P_j} + P_j) + (1-\mu)\ln\sqrt{P_i P_j}$$

$$P_i^2 \frac{\partial}{\partial P_i} O_{CGG}^{(d)}(P_i, P_j; \mu) = O_{CGG}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu P_i^2}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_j}{2\sqrt{P_i P_j}}\right) + \frac{(1-\mu)P_i^2 P_j}{2\sqrt{P_i P_j}}\right]$$

$$P_j^2 \frac{\partial}{\partial P_j} O_{CGG}^{(d)}(P_i, P_j; \mu) = O_{CGG}^{(d)}(P_i, P_j; \mu)\left[\frac{\mu P_j^2}{P_i - \sqrt{P_i P_j} + P_j}\left(1 - \frac{P_i}{2\sqrt{P_i P_j}}\right) + \frac{(1-\mu)P_j^2 P_i}{2\sqrt{P_i P_j}}\right]$$

$$P_i^{\,2} \frac{\partial}{\partial P_i} O_{CGG}^{(d)}\,(P_i, P_j\,;\mu) - P_j^{\,2} \frac{\partial}{\partial P_j} O_{CGG}^{(d)}\,(P_i, P_j\,;\mu)$$

$$= O_{CGG}^{(d)}\,(P_i, P_j\,;\mu)\left[\frac{\mu}{P_i - \sqrt{P_i P_j} + P_j}\left(P_i + P_j - \frac{\sqrt{P_i P_j}}{2}\right) + \frac{(1-\mu)}{2}\right](P_i - P_j)$$

Therefore, for $0 < P_i < P_j$,

$$(P_i - P_j)\left[P_i^{\,2} \frac{\partial}{\partial P_i} O_{CGG}^{(d)}\,(P_i, P_j\,;\mu) - P_j^{\,2} \frac{\partial}{\partial P_j} O_{CGG}^{(d)}\,(P_i, P_j\,;\mu)\right] \geq 0$$

By case 1 and case 2, $O_{CGG}\,(P_i, P_j\,;\mu)$ and its dual are Schur harmonic Convex.

4. Conclusion:

In this paper, some main properties of complementary geometric mean, Schur convexity, Schur geometric convexity and Schur harmonic convexity of oscillatory meansinvolving complementary geometric mean are discussed. Because of this feasibility, the researchers may use this to crack the problems pertaining to majorization, game theory and signal processing.As evidence, somespecial means satisfying these main properties are applied to remove noises in digital image [22].

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