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Law of Magnetic Fields Via Bianchi Identities

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Abstract

This article deal with the study of magnetic fields via bianchi identities. In the present study we discuss second order tensor for the magnetic flux and obtain the law of magnetic fields in the space time system w.r.t. to any frame of reference by using Bianchi identities of Riemannian Geometry. Finally, it notice that the time variant Maxwell's law of magnetic fields will include as a special case our results.

Keywords: *Maxwell's equations, magnetic fields, magnetic flux, Bianchi Identities, Riemannian geometry*

1. Introduction

Let M^n be a connected differential manifold of dimension $n > 2$ covered by system of co-ordinate neighbourhood (U, x^λ), where x^λindicate the local co-ordinates in U and the indices $\lambda = 1, 2, 3, \ldots\ldots n$and U isthe neighbourhood.

Let g be a second order tensor Riemannian metric with covariant component $g_{\lambda\mu}$ and contravariant component $g^{\lambda\mu}$.Let$V$ be the Riemannian connection with components$\Gamma^{\eta}_{\lambda\nu}$' called Riemann Christoffel symbols raising and lowering of indices are carried out by using $g_{\lambda\mu}$ and $g^{\lambda\mu}$ and Einstein summation conventions.

Let $R^{\sigma}_{\mu\nu\kappa}$ and $R_{\lambda\nu}$ be the Riemannian Christoffel curvature tensor field of type (1, 3) and Ricci curvature tensor of M^n respectively. Let r be the scalar curvature of space time system that is, (M^3, t), that is, M^4. We quote the following two famous identities from the Differential Geometry which is needed in our study (see [4], [6]).

$$R^{\lambda}_{\kappa\nu\mu} = -R^{\lambda}_{\nu\kappa\mu} \qquad (1.1)$$

$$\nabla_\sigma R^{\lambda}_{\kappa\nu\mu} + \nabla_\kappa R^{\lambda}_{\nu\sigma\mu} + \nabla_\nu R^{\lambda}_{\sigma\kappa\mu} = 0 \qquad (1.2)$$

In the Classical Differential Geometry, Bianchi [4] has proved second Bianchi identity. Many physicists and mathematicians established connections with the Bianchi identities. After that, Davies [5] used the Bianchi identities to find symmetric curvature to the gravitational field and curvature torsion relations.

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Recently, Bhati et al. [2] have investigated connection between continuity equations in Fluid Dynamics and the Bianchi identities. Moreover Bhati et al. [3] obtained law of electric fields by usingBianchi identities. Concurrent research avenues explored by the studies which were authored by Murali etc.al [7]-[24] delved into different forms, providing substantial insight into the nature of the work reported.

The main objective of this research article is to establish the law of magnetic fields in the space time system by using the Bianchi identities (1.1) and (1.2). In additionally, results of time variant Maxwell's law of magnetic fields in the cartesian frame were add as a special case.

We quote Maxwell's law given by Ahsan[1] and Davies [5] for the magnetic fields with respect to Cartesian frame of reference.

- If B is the time variant magnetic vector field, then Maxwell's law for magnetic fields in the differential form, locally at each point in the space
- $\nabla \cdot B = 0$ (1.3)
- The Maxwell's law of magnetic field in the integral form is given by:

$$\iint_S B \cdot n dS = 0 \tag{1.4}$$

Where B is the induction magnetic field vector on S,

n is the unit outward normal to the surface S and dS is the surface element of S.

Note: Proof of Maxwell's field equations solely depends on the Gauss's law of magnetic fields.

2. Mathematical Formulation

In this section, we consider the space time system, that is, $(M^3, t) = M^4$, the compact orientable Riemannian manifold M^4 with compact orientable boundary S and the magnetic flux tensor B. We quote the Stoke's theorem which is needed in this section.The indices λ, μ, ν, η running over the range $1, 2, 3, 4$. And one of the four co-ordinates may be taken as time co-ordinate say t. We study our result in space-time system and shown the importance of general theory of relativity (GRT) in Fluid Mechanics, theory of electricity, theory of magnetism in particular, Maxwell's magnetic fields equations and independent of the frame of reference. We quote the Stoke's Theorem from [6]

Stoke's Theorem: If B the tensor field of type (2,0) on M^4, then

$$\int_{M^4} \nabla_\nu B^{\lambda\nu}\, dV = \int_S B^{\lambda\nu} N_\nu\, dS \tag{2.1}$$

where N_ν is the components of the unit outward normal to the boundary S, $B^{\lambda\nu}$ is the contravariant components of tensor B, dS is the surface element of S and dV is the volume element of M^4.

In this section, we prove the following law for magnetic fields in M^4 which are true in any frame of reference.

I. Law of fields:

If B is the magnetic field tensor with components $B^{\mu\nu}$ as

If B is the magnetic field tensor with components $B^{\mu\nu}$ as

$$B^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} r g^{\mu\nu} \qquad (2.2)$$

then locally at each point in space,

$$\nabla_\nu B^{\mu\nu} = 0 \qquad (2.3)$$

The equation (2.3) simply asserts that the divergence of magnetic fields in space time system in any frame of reference is zero.

II. Law for magnetic field in the integral form:

If B is the tensor field of type (2, 0) in M^4, then law in

the integral form is given by

$$\int_S B^{\lambda\nu} N_\nu dS = 0 \qquad (2.4)$$

where N_ν is the components of the unit outward normal to the boundary, $B^{\lambda\nu}$ is the contravariant components of the tensor field B, dS is the surface element of S.

The equation (2.4) simply asserts that magnetic flux acrss the closed surface S is zero in the space time system with respect to any frame of reference.

3. Method of solution

Proof of I: Law of magnetic fields:

From equation (1.1) and equation (1.2) and contracting equation (1.2) with respect to λ and σ, we get

$$\nabla_\eta R^\eta_{\kappa\nu\mu} - \nabla_\kappa R_{\nu\mu} + \nabla_\nu R_{\kappa\mu} = 0$$

Now multiplying by $g^{\nu\mu}$, we get

$$\nabla_\sigma R^\sigma_k = \frac{1}{2} \nabla_k r$$

Again, multiplying by $g^{\lambda\kappa}$, we obtain

$$\nabla_\sigma \left(g^{\lambda\kappa} R^\sigma_\kappa\right) = \frac{1}{2} g^{\lambda\kappa} \nabla_\kappa r$$

$$\nabla_\sigma R^{\lambda\sigma} = \frac{1}{2} \nabla^\lambda r \qquad (2.5)$$

where

$\nabla^\lambda = g^{\lambda\kappa} \nabla_\kappa$ and $g^{\lambda\kappa} R^\sigma_\kappa = R^{\lambda\sigma}$

From (2.2), we have the matrix:

$$\left(B^{\lambda\eta}\right) = \begin{bmatrix} B^{11} & B^{12} & B^{13} & B^{14} \\ B^{21} & B^{22} & B^{23} & B^{24} \\ B^{31} & B^{32} & B^{33} & B^{34} \\ B^{41} & B^{42} & B^{43} & B^{44} \end{bmatrix}$$

Equation (2.2) gives 16 equations. Since $B^{\nu\lambda} = B^{\lambda\nu}$, *i.e.B* is symmetric for λ, μ and the independent components are 10. Thus equation (2.2) yields 10 equations relatively than 16 equations. Each equation is given corresponding to the following each component from equation (2.2), that is,

$$B^{11}, B^{22}, B^{33}, B^{44}, B^{12}, B^{13}, B^{14}, B^{23}, B^{24}, B^{34}$$

From (2.2), taking covariant derivative with respect to ν, we get

$$\nabla_\nu B^{\mu\nu} = \nabla_\nu \left[R^{\mu\nu} - \frac{1}{2} r g^{\mu\nu}\right]$$

$$= \left[\nabla_\nu R^{\mu\nu} - \frac{1}{2} \nabla_\nu r g^{\mu\nu}\right]$$

$$= \frac{1}{2}\nabla^\mu r - \frac{1}{2}\nabla^\mu r$$

$$= 0$$

where in, using equation (2.2), we obtained:

$$\nabla_\nu B^{\mu\nu} = 0, \qquad \lambda, \mu = 1, 2, 3, 4 \quad (2.6)$$

Which is the equation (2.2) and gives the required law of magnetic field locally at each point in space. This completes proof of I : Law of magnetic fields in the differential form.

Analysis of the equation (2.6):

From the equation (2.6), for $\nu = 1$,

$$B^{\mu 1} + \nabla_2 B^{\mu 2} + \nabla_3 B^{\mu 3} + \nabla_4 B^{\mu 4} = 0, \mu = 1, 2, 3, 4 \qquad (2.7)$$

Thus, we have for $\mu = 1, 2, 3, 4$

$$\nabla_1 B^{11} + \nabla_2 B^{12} + \nabla_3 B^{13} + \nabla_4 B^{14} = 0 \qquad (2.8)$$

$$\nabla_1 B^{21} + \nabla_2 B^{22} + \nabla_3 B^{23} + \nabla_4 B^{24} = 0 \qquad (2.9)$$

$$\nabla_1 B^{31} + \nabla_2 B^{32} + \nabla_3 B^{33} + \nabla_4 B^{34} = 0 (2.10)$$

$\nabla_1 B^{41} + \nabla_2 B^{42} + \nabla_3 B^{43} + \nabla_4 B^{44} = 0$ (2.11)

From the equation (2.7), for instance, covariant differentiation of B with respect to x^1 is given by

$$\nabla_1 B^{1\nu} = \frac{\partial B^{1\lambda}}{\partial x^1} + g^{\mu 1}\Gamma^{\lambda}_{\mu 1} + g^{1\mu}\Gamma^{\lambda}_{1\mu}$$

$$\nabla_1 B^{11} = \frac{\partial B^{11}}{\partial x^1} + 2(g^{11}\Gamma^{1}_{11} + g^{21}\Gamma^{1}_{21} + g^{31}\Gamma^{1}_{31} + g^{41}\Gamma^{1}_{41}), \quad (\lambda = 1)$$

Similarly, for other covariant derivatives of B with respect to x^2, x^3 and x^4 in (2.7), (2.8), (2.9), (2.10) and (2.11) may be given.

Any single solution $g^{\lambda\nu}(x^{\lambda})$ requires the 4-parameters family of solution which is obtained simply as $g^{\lambda\nu}((x^{l})^{\lambda}$ induced by a ordinary co-ordinate transformations. Choosing a specific co-ordinate system, this singles out a unique solution among this family.

Proof of II: Law of magnetic fields in the integral form

If B with components $B^{\lambda\nu}$ is the magnetic flux tensor on M^4, then from Stoke's theorem, that is, from (2.1), we have

$\int_S B^{\lambda\nu} N_\nu \, dS = \int_{M^4} \nabla_\nu B^{\lambda\nu} \, dV = 0$ (2.12)

Where in, using equation (2.6), we get the required equation (2.4). This completes the proof.

4. Results and Discussions

We obtain the law of magnetic fields in the space time system *w.r.t.* any frame of reference by using Bianchi identities and not using Gauss's law of magnetic fields.

For the Cartesian frame of reference, equations (2.6) and (2.12) can be reduced to the equations (1.3) and (1.4) respectively. Thus, our results are the natural generalization of Maxwell's magnetic field equations (1.3) and (1.4).

5. Conclusions

We obtain the law of magnetic fields in the space time system *w.r.t.*any frame of reference by by using Bianchi identities and not using Gauss's law of magnetic fields.

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