Numerical Finite Difference Scheme For A Two-Dimensional Time-Fractional Semi Linear Partial Differential Equations

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Abstract: The fundamental tools for modelling neural dynamics are time-fractional partial differential equations. In order to solve a two-dimensional, time-fractional semilinear parabolic equation under Dirichlet boundary conditions, this paper introduces the Crank-Nicolson (C.N.) finite difference methodology. The proposed scheme's consistency, stability, and convergence are also thoroughly investigated. Two numerical experiments are conducted to support the theoretical results. The effectiveness of the method is carefully assessed and analysed in terms of absolute mistakes, order of accuracy, and computational time. The outcomes show that, while being conditionally stable, the suggested scheme may be used successfully with a high rate of convergence to calculate numerical solutions for the issue at hand.
Keywords: Two-Dimensional, Time-Fractional, Semi Linear, Partial Differential Equations

1. Introduction:

Fractional calculus, a branch of mathematics concerned with the properties of derivatives and integrals of non-integer orders, has been an area of interest since the inception of classical calculus. This mathematical field specializes in the resolution of time-dependent fractional differential equations, which entail fractional derivatives. Regarded as a pivotal tool for the exploration of dynamical systems, fractional calculus is valued for its non-local operators that encapsulate the historical progression of dynamics. The usage of fractional calculus and fractional processes has become a favored approach when grappling with the unique properties of long-term memory effects found in multiple domains of applied sciences. These fields range from finance and economic dynamics, biological systems and bioinformatics, nonlinear waves and acoustics, to image and signal processing, transportation systems, geosciences, astronomy, and cosmology

Since the beginning of classical calculus, there has been interest in the field of fractional calculus, which deals with the characteristics of derivatives and integrals of non-integer orders. This branch of mathematics specialises in the solution of fractional derivative-based time-dependent differential equations. Fractional calculus is prized for its non-local operators that capture the development of dynamics across time, making it a crucial tool for the investigation of dynamical systems. The use of fractional calculus and fractional processes has grown to be a preferred strategy when addressing the distinctive characteristics of long-term memory effects seen in numerous applied science disciplines. These disciplines include nonlinear waves and acoustics, biological systems and bioinformatics, finance and economic dynamics, image and signal processing, and Numerous researchers, including but not limited to Heaviside, Lagrange Riemann, Liouville, Grunwald, Euler, Fourier, and Abel, have gradually added to the development of fractional calculus. The popularity of fractional calculus nowadays can be attributed to the differ integral operator's versatility, which combines both integer-order derivatives and integrals as special instances. As explained by Podlubny and Kisela, the fractional integral, for example, can be used to depict the accumulation of a quantity when the order of integration is unknown and can be inferred as a parameter of a regression model. On the other hand, damping is typically represented by the fractional derivative. Other applications include viscoelasticity, electrical networks, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, rheology, optics and signal processing, fluid flow, diffusive transport, probability and signal processing, transportation systems, geosciences, astronomy, and cosmology statistics. Numerous analytical and numerical techniques have been applied in recent years to solve fractional differential equations. The Fourier transformation method, the Laplace transformation method, and the green function method are examples of analytical approaches. However, analytical solutions are rarely found for fractional differential equations. Therefore, it is essential to create numerical schemes for these equations. Time (space) fractional differential equations have been successfully solved using techniques including the finite difference approach, the spectral method, and the finite element method. The goal of this work is to suggest a high order efficient numerical technique for a two-dimensional semi-linear time fractional equation with Dirichlet boundary conditions because most mathematical models that describe real-world phenomena involve non-linear fractional partial differential equations and the majority of prior studies have concentrated on linear types. Specifically, using mathematical induction, we propose the linearly implicit Crank-Nicolson finite difference scheme and demonstrate its consistency, conditional stability, and convergence. The remainder of this essay is structured as follows: In section two, we give a literature overview on the subject under study as well as the mathematical formulation of the temporal fractional diffusion equation. The finite difference scheme's derivation, consistency, stability, and convergence are discussed in section four of section three. In the final section, some conclusions are provided.

2. Two-Dimensional Fractional Semilinear Diffusion Equation

Take into consideration the fractional semi-linear diffusion equation in two dimensions of space and time.

$$\frac{\partial^a u(x,y,t)}{\partial t^a} = a(x,y,t)D_x^{\beta}u(x,y,t) + b(x,y,t)D_y^{\gamma}u(x,y,t) + f(x,y,t,u) \quad (1)$$

$$0 < x < L_x, 0 < y < L_y, 0 < t \le T$$

$$with\ initial\ condition\ u(x,y,0) = g(x,y) \quad (2)$$

Boundary condition

$$u(0,y,t) = 0 = u(L_x,y,t) \quad (3)$$

$$u(x,0,t) = 0 = u\left(x,L_y,t\right) \quad (4)$$

Positive functions are represented by the diffusion coefficients *a(x, y, t) and b(x, y, t)*, respectively. For the two-dimensional space-time fractional semi-linear diffusion equation, this model is referred to as the first IBVP.

2.1 Implicit Finite Difference Scheme

Discerning the first IBVP (1) is the focus of this section (4). Define $t_k = k\tau, \mathrm{k} = 0,1,2, \ldots \mathrm{n}; x_i = i\Delta x, i = 0,1,2,\ldots,l, \mathcal{Y}_j = j\Delta y, j = 0,1,2,\ldots,m, where\ \tau = \frac{T}{n}, \Delta x = \frac{L_x}{l}, \Delta\mathcal{Y} = \frac{L_{\mathcal{Y}}}{m}$ are progressions in time and space, respectively. Let $u_{i,j}^k$ be the approximate value that can be obtained using numbers to $u(x_i,\mathcal{Y}_j,t_k) and\ f_{i,j}^k(u_{i,j}^k) = f\left(x_i,\mathcal{Y}_j,t_k,u(x_i,\mathcal{Y}_j,t_k)\right)$ *and the approximation looks like this:*

$$f\left(x_i,\mathcal{Y}_j,t_{k+1},u(x_i,\mathcal{Y}_j,t_k)\right) = f\left(x_i,\mathcal{Y}_j,t_k,u(x_i,\mathcal{Y}_j,t_k)\right) + O(t^a) \quad (5)$$

Take it further that the nonlinear function $f_{i,j}^k(u_{i,j}^k)$ is fulfilling the Lipschitz criterion.

$$\left|f_{i,j}^k(\bar{u}_{i,j}^k) - f_{i,j}^k(u_{i,j}^k) = \right|\bar{u}_{i,j}^k - u_{i,j}^k| = | \le L\left|\epsilon_{i,j}^k\right| \quad (6)$$

Following is an approximation of the time fractional derivatives that we compute here.

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{k} b_s\delta_t u\left(x_i,y_j,t_{k-s}\right) + O(\tau) \quad (7)$$

Where $b_s = (s+1)^a - s^a, s = 0,1,2,\ldots,n\ and\ \delta_t$ is the operator of the forward difference? In order to discretize the space fractional Riemann–Liouville derivatives $D_x^{\beta}u(x,\mathcal{Y},t) and\ D_{\mathcal{Y}}^{\gamma}u(x,\mathcal{Y},t)$. Following is an example of how we implemented the shifted Grunwald formula at level t_{k+1}.

$$D_x^{\beta} u(x_i, y_j, t_{k+1}) = \frac{1}{h^{\beta}} \sum_{p=0}^{i+1} g_p^1 u(x_{i+1-p}, y_j, t_{k+1}) + O(h) \quad (8)$$

$$D_y^{\gamma} u(x_i, y_j, t_{k+1}) = \frac{1}{h^{\gamma}} \sum_{q=0}^{j+1} g_q^2 u(x_i, y_{j+1-q}, t_{k+1}) + O(h)$$

Where

$$g_0^1 = 1, g_p^1 = (-1)^p \frac{\beta(\beta-1)(\beta-2)\ldots(\beta-p+1)}{p!}, p = 1,2,3, \ldots$$

$$g_0^2 = 1, g_q^2 = (-1)^q \frac{\gamma(\gamma-1)(\gamma-2)\ldots(\gamma-q+1)}{q!}, q = 1,2,3, \ldots \quad (9)$$

The discretization of the first IBVP occurs as follows:

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{k} b_s \delta_t u(x_i, y_j, t_{k-s}) = \frac{1}{h^{\beta}} \sum_{p=0}^{i+1} g_p^1 u(x_{i+1-p}, y_j, t_{k+1}) + \frac{1}{h^{\gamma}} \sum_{q=0}^{j+1} g_q^2 u(x_i, y_{j+1-q}, t_{k+1}) + f_{i,j}^k(u_{i,j}^k)$$

Regarding the rearrangement, we have –

$$\sum_{s=0}^{k} b_s \delta_t u(x_i, y_j, t_{k-s}) = r_1 \sum_{p=0}^{i+1} g_p^1 u(x_{i+1-p}, y_j, t_{k+1}) + r_2 \sum_{q=0}^{j+1} g_q^2 u(x_i, y_{j+1-q}, t_{k+1}) + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^k(u_{i,j}^k) \quad (10)$$

Where

$$\mu_1 = \frac{\tau^{\alpha}}{h^{\beta}}, r_1 = r_1(i,j,k) = \mu\Gamma(2-\alpha) a_{i,j}^{k+1}$$

$$\mu_2 = \frac{\tau^{\alpha}}{h^{\gamma}}, r_2 = r_2(i,j,k) = \mu\Gamma(2-\alpha) b_{i,j}^{k+1}$$

$$\delta_t u_{i,j}^k = u_{i,j}^{k+1} - u_{i,j}^k$$

$for\ i = 0, 1, 2, \ldots, l-1;\ j = 0, 1, 2, \ldots m-1;\ k = 0, 1, 2, \ldots n-1. Let$

$$Lu_{i,j}^{k+1} = u_{i,j}^{k+1} - r_1 \sum_{p=0}^{i+1} g_p^1 u_{i+1-p,j}^{k+1} - r_2 \sum_{q=0}^{i+1} g_q^1 u_{i,j-q+1}^{k+1}$$

We get

$$Lu_{i,j}^{k+1} = u_{i,j}^k - \sum_{s=1}^{k} b_s u_{i,j}^{k+1-s} + \sum_{s=1}^{k} b_s u_{i,j}^{k-s} + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^k(u_{i,j}^k)$$

Hence, for k = 0;

$$Lu_{i,j}^1 = u_{i,j}^0 + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^0(u_{i,j}^0) \quad (11)$$

for k > 0;

$$Lu_{i,j}^{k+1} = b_k u_{i,j}^0 - \sum_{s=0}^{k-1} (b_s - b_{s+1}) u_{i,j}^{k-s} + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^k(u_{i,j}^k) \quad (12)$$

As a result, the full discrete form of the first IBVP, which ranges from 1 to 4, is

$$Lu_{i,j}^1 = u_{i,j}^0 + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^0(u_{i,j}^0)$$

$$Lu_{i,j}^{k+1} = b_k u_{i,j}^0 - \sum_{s=0}^{k-1} (b_s - b_{s+1}) u_{i,j}^{k-s} + \tau^{\alpha}\Gamma(2-\alpha) f_{i,j}^k(u_{i,j}^k) \quad (13)$$

$$initial\ condition\ u_{i,j}^0 = \phi_{i,j}$$

Boundary conditions

$$u_{0,j}^k = 0 = u_{l,j}^k \quad (14)$$

$$u_{i,0}^k = 0 = u_{i,m}^k$$

$$Let\ Uk = \begin{pmatrix} u_1^k \\ u_2^k \\ \cdot \\ u_{1-1}^k \end{pmatrix}, F^k = \begin{pmatrix} f_1^k \\ f_2^k \\ \cdot \\ \cdot \\ \cdot \\ f_{1-1}^k \end{pmatrix}, \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \cdot \\ \cdot \\ \Phi_{1-1} \end{pmatrix}$$

$$\text{where } \mathbf{u_i^k} = \begin{pmatrix} u_{i,1}^k \\ u_{i,2}^k \\ \cdot \\ \cdot \\ u_{i,m-1}^k \end{pmatrix}, \mathbf{f_i^k} = \begin{pmatrix} f_{i,1}^k(u_{i,1}^k) \\ f_{i,2}^k(u_{i,2}^k) \\ \cdot \\ \cdot \\ \cdot \\ f_{i,m-1}^k(u_{i,m-1}^k) \end{pmatrix}, \mathbf{\Phi_i} = \begin{pmatrix} \phi_{i,1} \\ \phi_{i,2} \\ \cdot \\ \cdot \\ \phi_{i,m-1} \end{pmatrix}$$

$i = 1,2,\dots,l-1;\ k = 0,1,2,\dots,n$

The equation that was just presented can also be represented in matrix form.

$$\begin{cases} AU^1 = U^0 + \tau^\alpha \Gamma(2-\alpha)F^1 \\ AU^{k+1} = \sum_{j=0}^{k-1} (b_j - b_{j+1})U^{k-j} + b_k U^0 + \tau^\alpha \Gamma(2-\alpha)F^{k+1} \\ U^0 = \Phi \end{cases} \tag{15}$$

Where$A = [A_{i,j}]$ is a matrix containing all of the coefficients? Following is a statement that we derive from the equation (15).

2.2 Finite Difference Scheme

In equation, the discretization of the second order spatial derivative is accomplished by applying the difference formula-

$$u_{xx} = \frac{u(x_{l-1},t_{k+1}) - 2u(x_l,t_{k+1}) + u(x_{l+1},t_{k+1})}{2h^2} + \frac{u(x_{l-1},t_k) - 2u(x_l,t_k) + u(x_{l+1},t_k)}{2h^2} O(h^2) \tag{16}$$

Combining equations

$$-r_l^{k+1}u_{l-1}^{k+1} + (1+2r_l^{k+1})u_l^{k+1} - r_l^{k+1}u_{l+1}^{k+1} = r_l^{k+1}u_{l-1}^k + (1-2r_l^{k+1})u_l^k + r_l^{k+1}u_{l+1}^k - \sum_{j=1}^{k} (u_l^{k+1-j} - u_l^{k-j})b_j^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})f_l^k(u_l^k)$$

The Crank-Nicolson finite difference approach is therefore appropriate for the first initial boundary value problem

$$-r_l^{k+1}u_{l-1}^{k+1} + (1+2r_l^{k+1})u_l^{k+1} - r_l^{k+1}u_{l+1}^{k+1} = r_l^{k+1}u_{l-1}^k + (1-2r_l^{k+1})u_l^k + r_l^{k+1}u_{l+1}^k - \sum_{j=1}^{k} (u_l^{k+1-j} - u_l^{k-j})b_j^{l,k+1} + \tau_l^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})f_l^k(u_l^k) \tag{17}$$

$$initial\ condition\ u_l^0 = g(x_l), l = 0,1,2,\dots,M \tag{18}$$

$$and\ boundaryu\ conditions\ u_0^k = 0 = u_M^k, k = 0,1,2,\dots,N \tag{19}$$

3. Conclusion:

In this paper, the Crank-Nicolson numerical finite difference scheme is proposed to solve a two-dimensional time-fractional Semilinear parabolic equation with homogeneous Dirichlet boundary conditions. We prove that the proposed scheme is consistent, conditionally stable and convergent. In addition, in order to support the theoreticalIn this paper, the Crank-Nicolson numerical finite difference scheme is proposed to solve a two-dimensional time-fractional Semilinear parabolic equation with homogeneous Dirichlet boundary conditions. We prove that the proposed scheme is consistent, conditionally stable and convergent. In addition, in order to support the theoretical

In this study, a two-dimensional time-fractional semi linear parabolic equation with homogeneous Dirichlet boundary conditions is solved using the Crank-Nicolson numerical finite difference method. The weighted average finite difference scheme is more general scheme in the study of nonlinear partial differential equations. Explicit, implicit and Crank-Nicolson finite difference schemes are the particular cases of weighted average finite difference scheme. We demonstrate the consistency, conditional stability, and convergence of the suggested scheme. Additionally, two numerical experiments are taken into consideration to corroborate the theoretical findings. The nonlinearity that is being considered is a polynomial of the third order with three different roots. The Hirota method allows one to arrive at the correct answer in any given situation. After that, this one-of-a-kind solution may be put to work in the development of nonstandard discrete models. It should be noted, however, that exact-finite difference schemes for partial differential equations are not expected to exist. A comparison will be made between the partial differential equation under consideration and the standard finite-difference schemes. This comparison will focus on how the solutions of the various nonstandard and standard discrete models differ from one another.

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