Numerical Analysis of Singularly Perturbed System of parabolic convection-diffusion IBVP system with boundary layers

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Abstract: In this paper, we get the numerical solution of a singularly perturbed system of boundary layer-exhibiting parabolic convection-diffusion problems. The backward-Euler method and an upwind finite difference scheme make up the suggested numerical approach for the time and spatial derivatives, respectively. For the spatial discretization, we analyse the scheme on a piecewise uniform Shishkin mesh in order to establish uniform convergence with regard to the perturbation parameters. The stability analysis for the suggested method is provided, and a parameter-uniform error estimate is generated. We have conducted some numerical tests to verify the theoretical findings.

Keywords: Numerical Analysis, Singularly Perturbed System, boundary layers

1. Introduction

In this section, we use the difference approach suggested in the previous section to solve the semi-linear parabolic CD IBVP system:

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} - \varepsilon \frac{\partial^2 \vec{u}}{\partial x^2} - A(x)\frac{\partial \vec{u}}{\partial x} + \vec{f}(x,t,\vec{u}) = \vec{0}, (x,t)\epsilon Q \\ \vec{u}(x,0) = \overrightarrow{u_0}(x), x\epsilon\overline{\Omega_x} \\ \vec{u}(0,t) = \vec{0}, \vec{u}(1,t) = \vec{0}, t\epsilon(0,T] \end{cases} \quad (1)$$

Where $\vec{f}(x,t,\vec{u}) = (f_1(x,t,u_1,u_2), f_2(x,t,u_1,u_2))^T$. We're going to go ahead and presume $f_1\, and f_2$ these operations are sufficiently slick. In addition, we can suppose

$$\frac{\partial f_1}{\partial u_1} \geq \gamma_\beta > 0, \frac{\partial f_2}{\partial u_2} \geq \gamma_\beta > 0, \frac{\partial f_1}{\partial u_2} < 0, \frac{\partial f_2}{\partial u_1} < 0. \text{ in } \bar{\Omega}_x \times [0,T] \times \mathbb{R}^2 \quad (2)$$

$$min\left\{\frac{\partial f_1}{\partial u_1} + \frac{\partial f_1}{\partial u_2} \cdot \frac{\partial f_2}{\partial u_1} + \frac{\partial f_2}{\partial u_2}\right\} \geq \tilde{\beta} > 0, \text{ in } \bar{\Omega}_x \times [0.T] \times \mathbb{R}^2 \quad (3)$$

The combination of these constraints (1) and (2), as well as the implicit function theorem, ensures that there is only one solution. $\vec{u}(x,t) \in (C_0^2(\bar{Q}))^2$ in addition to the problem (3), the solution $\vec{u}(x,t)$ Contains boundary layers that overlap along the line x = 0.

First, we shall linearize the semi-linear problem using Newton's quasi-linearization approach (3), and then we will use this sequence of linear problems to solve the other problems. Whose solutions are we talking about $\vec{u}^P(x,t)$ when you make a good first guess $\vec{u}^0(x,t)$ converge on a single, correct answer u This linear parabolic IBVP is solved:

$$\begin{cases} \frac{\partial \vec{u}^{p+1}}{\partial t} - \mathcal{E}\frac{\partial^2 \vec{u}^{p+1}}{\partial x^2} - A(x)\frac{\partial \vec{u}^{p+1}}{\partial x} + \mathcal{J}\vec{u}^{p+1} = \mathcal{F}(x,t,u_1^p,u_2^p), (x,t) \in Q \\ \vec{u}(x,0) = \vec{u}_0(x), x \in \bar{\Omega}_x, \\ \vec{u}(0,t) = \vec{0}, \vec{u}(1,t) = \vec{0}. t \in (0,T]. \end{cases} \quad (4)$$

Here is the Jacobian matrices, with the reaction coefficient J

$$\mathcal{J}(x,t)=\begin{pmatrix}\frac{\partial f_1(x,t,u_1^p,u_2^p)}{\partial u_1} & \frac{\partial f_1(x,t,u_1^p,u_2^p)}{\partial u_2}\\ \frac{\partial f_2(x,t,u_1^p,u_2^p)}{\partial u_1} & \frac{\partial f_2(x,t,u_1^p,u_2^p)}{\partial u_2}\end{pmatrix}$$

and the source term $\mathcal{F}(x,t,u_1^p,u_2^p)=\mathcal{J}(x,t)\vec{u}^p-\vec{f}(x,t,u_1^p,u_2^p)$.. Given in (3) -(4), Therefore, it is safe to say that $\mathcal{J}(x,t)$is an L_{0-1} matrix.

For$p\in\mathbb{N}$, Take into consideration the following stipulations for stopping

$$|\vec{u}^{p+1}(x_i,t_n)-\vec{u}^p(x_i,t_n)|\le Tol,\text{ for }(x_i,t_n)\in Q^{N,M},p\ge 0.$$

Here,$T\ ol$ denotes the tolerance bound selected by the user The numerical findings of a system of semilinear IBVP of the forms (3) and (4) are reported in the final portion of this chapter.

Example: Consider the semi-linear parabolic IBVP system shown below. Q = (0; 1)× (0; 1]:

We obtain the following singular perturbation system of linear parabolic IBVP by using Newton's linearization procedure (3) to Example:

$$\frac{\partial u_1}{\partial t}-\varepsilon_1\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial u_1}{\partial x}+\exp(u_1-u_2)=0,$$
$$\frac{\partial u_2}{\partial t}-\varepsilon_2\frac{\partial^2 u_2}{\partial x^2}-\frac{\partial u_2}{\partial x}+\exp(u_2-u_1)=0,$$
$$u_1(x,0)=\frac{1-\exp(-x/\varepsilon_1)}{1-\exp(-1/\varepsilon_1)}-x,u_2(x,0)=\frac{1-\exp(-x/\varepsilon_2)}{1-\exp(-1/\varepsilon_2)}-x,$$
$$u_1(0,t)=u_1(1,t)=0,u_2(0,t)=u_2(1,t)=0,t\in[0,1]$$

$$\begin{cases}\frac{\partial u_1^{p+1}}{\partial t}-\varepsilon_1\frac{\partial^2 u_1^{p+1}}{\partial x^2}-\frac{\partial u_1^{p+1}}{\partial x}+\exp(u_1^p-u_2^p)u_1^{p+1}-\exp(u_1^p-u_2^p)u_2^{p+1}\\ \frac{\partial u_2^{p+1}}{\partial t}-\varepsilon_2\frac{\partial^2 u_2^{p+1}}{\partial x^2}-\frac{\partial u_2^{p+1}}{\partial x}-\exp\left(u_2^p-u_1^p\right)\Big)u_1^{p+1}+\exp\left(u_2^p-u_1^p\right)\Big)u_2^{p+1}\\ =-\exp(u_2^p-u_1^p)u_1^p+\left(\exp\left(u_2^p-u_1^p\right)-1\right)u_2^p,\\ u_1^{p+1}(x,0)=u_1(x,0),u_2^{p+1}(x,0)=u_2(x,0),\\ u_1^{p+1}(0,t)=u_1^{p+1}(1,t)=0,u_2^{p+1}(0,t)=u_2^{p+1}(1,t)=0,t\in[0,1].\end{cases}$$

As a result, we solve the preceding linearized problem for a fixed p using the computational method presented in Section. We employ the Newton's linearization procedure as a convergence criterion.

$$max\left\{\left|U_{1,i}^{n(p)}-U_{1,i}^{n(p-1)}\right|,\left|U_{2,i}^{n(p)}-U_{2,i}^{n(p-1)}\right|\right\}\le 10^{-7}$$

where, we choose $U_{1,i}^{n(0)}-U_{2,i}^{n(0)}=0$as a starting point When we hit the required tolerance bound, we stop iterating and consider the problem solved.

Because the precise solution of Example is also unknown, we will employ the previously described double-mesh technique to get the precision of the numerical solution as well as to demonstrate the E-uniform convergence of the suggested scheme.

Demonstrate the numerical solution to Example, which includes boundary layers. The depicts the monotonically declining behavior of "-uniform errors computed for N = 32; 64; 128; 256; 512, and M = N with the singular perturbation parameters. $\hat{S}_e=\{(\varepsilon_1,\varepsilon_2)\mid(2^{-18},2^{-8}),(2^{-20},2^{-10})\}$. The uniform error behaviour may also be seen in the loglog.

Remark : On account of condition, the arrangement of the limit esteem issue fulfils the measure

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$$|\mathbf{u}(a,t)| \le 2(1-m^2)^{-1} max\left[c_0^{-1} \max_{\bar{G}} |\mathbf{g}(a,t,\mathbf{0})|, \max_{S} |\boldsymbol{\varphi}(a,t)|\right], (a,t) \in \bar{G}$$

Here m = $m_{(4.3)}$. For the component u^i (x, t) we get the estimation

$$\left|u^i(a,t)\right| \le m \max_{\bar{G}} \left|u^{3-i}(a,t)\right| + c_0^{-1} \max_{\bar{G}} \left|g^i(a,t,\mathbf{0})\right| + \max_{S} \left|\varphi^i(a,t)\right|,$$

$$(a,t) \in \bar{G}, i = 1,2$$

In the scenario of linear equations, we now provide estimates derived from the primary terms of the solution's asymptotic expansion. First, we write the problem's answer as the sum of functions.

$$u(a,t) = U(a,t) + V(a,t), (a,t) \in \bar{G} \qquad (5)$$

where U(a, t) and V(a, t) are the solution decomposition's regular and singular terms. The function U(a, t), (a, t) $\in \bar{G}$ is the restriction to $\bar{G}$ of the function $U^e(a,t), (a,t) \in \bar{G}^e$, *where the set* $\bar{G}^e$, *i.e.*, the extension of $\bar{G}$ beyond the boundary $\bar{S}^L$ includes $\bar{G}$ along with its m_0-neighbourhood; $\bar{G}^e = \bar{D}^e \times [0,T]$. The function $U^e(a,t)$ is the solution of the problem

$$\begin{aligned} &L^e\mathbf{U}^e(a,t) = \mathbf{g}^e(a,t,\mathbf{U}^e(a,t)), && (a,t) \in G^e, \\ &\mathbf{U}^e(a,t) = \varphi^e(a,t), && (x,t) \in S^e \end{aligned} \qquad (6)$$

Here L^e *and* $g^e(a,t,u), (x,t) \in \bar{Q}$ are smooth continuations of the operator $L_{(4.2)}$ and the function g(a, t, u) the function $\varphi^e(a,t), (a,t) \in S^e$ is chosen sufficiently smooth $\varphi^e(a,t) = \varphi(a,t), (a,t) \in S_0$. Assume that the function $g^e(a,t,u)$ *and* $\varphi^e(a,t)$ are equal to zero outside a nearest m_1-neighbourhood of the set $\bar{G}$ where $m_1 < m_0$. The solution to the problem is the function V(a, t)

$L_{(4.2)}$ V(a, t) = g (a, t, U(a, t) + V(a, t)) − g (a, t, U(a, t)) , (a, t) ∈ G, V(a, t) = ϕ(a, t) − U(a, t) ≡ ϕ_V(a, t), (a, t) ∈ (7)

In the representation, we now estimate the regular component of the problem solution (7). Let's write U(a, t) as the sum of two functions.

$$\mathbf{U}(a,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}_k(a,t) + \mathbf{v}_{\mathbf{U}}^n(a,t) \equiv \mathbf{U}^n(a,t) + \mathbf{v}_{\mathbf{U}}^n(a,t), (a,t) \in \bar{G}, \qquad (8)$$

that corresponds to the function's representation $\mathbf{U}^e(a,t), (a,t) \in \bar{G}^e$, Which of the following is the solution to problem (7):

$$\mathbf{U}^e(a,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{U}_k^e(a,t) + \mathbf{v}_{\mathbf{U}}^{en}(a,t), (a,t) \in \bar{G}^e$$

The function $U_k^e(a,t), (a,t) \in \bar{G}^e$, i.e, problem solutions are components in the expansion of the regular part of the solution.

$$L_{(4.14)}U_0^e(a,t) = g^e\big(a,t, U_0^e(a,t)\big), \qquad (a,t) \in \bar{G}^e/S_0^e \qquad (9)$$

$$U_0^e\,(a,t) = \varphi^e(a,t), \qquad (a,t)\epsilon S_0^e;$$

$$L_{(4.14)}U_k^e(a,t) = \varepsilon^{-2}\{L_{(4.14)} - L_{(4.11)}^e\}U_{k-1}^e(a,t)$$

$$+\varepsilon^{-2k}\left\{\mathbf{g}^e\left(a,t,\sum_{k_1=0}^{k}\varepsilon^{2k_1}\mathbf{U}_{k_1}^e(a,t)\right) - \mathbf{g}^e\left(a,t,\sum_{k_1=0}^{k-1}\varepsilon^{2k_1}\mathbf{U}_{k_1}^e(a,t)\right)\right\},$$

$$(a,t) \in \bar{G}^e \setminus S_0^e$$

$$\mathbf{U}_k^e(a,t) = \mathbf{0}, (a,t) \in S_0^e, k > 0$$

Where

$$L_{(4.14)} - L_{(4.11)}^e\big|_{\varepsilon=0} = -C^e(a,t) - P^e(a,t)\frac{\partial}{\partial t}$$

For the function $v_U^{en}(a,t)$ The following is our estimate:

$$|\mathbf{v}_{\mathbf{U}}^{en}(a,t)| \le M\varepsilon^{2n+2}, (a,t) \in \bar{G}$$

If a condition exists, (4.8), where

$$l \ge K-2,\; l_1 \ge K+2n, \qquad (10)$$

For

$$n = [(K+1)/2]_{3.2} - 2, \qquad K \geq 4 \qquad (11)$$

One has $\mathbf{U}^e \in H^{K+\alpha}(\bar{G}^e)$ We derive the estimate for the function U (a, t)

$$\left|\frac{\partial^{k+k_0}}{\partial a_1^{k_1}\partial a_2^{k_2}\partial t^{k_0}}\mathbf{U}(a,t)\right| \leq M[1+\varepsilon^{K-k-2}], (a,t) \in \bar{G}, k+2k_0 \leq K. \quad (12)$$

Furthermore, for the components $U^n(x,t) and\ v_U^n(x,t)$ we have the estimations

$$\left|\frac{\partial^{k+k_0}}{\partial a_1^{k_1}\partial a_2^{k_2}\partial t^{k_0}}\mathbf{U}^n(a,t)\right| \leq M,$$
$$\left|\frac{\partial^{k}}{\partial a_1^{k_1}\partial a_2^{k_2}\partial t^{k_0}}\mathbf{v}_{\mathbf{U}}^n(a,t)\right| \leq M\varepsilon^{K-k-2}, (a,t) \in \bar{G}, k+2k_0 \leq K.$$

Remark. The function, according to the decomposition (4.13), $\varphi_{v(4.12)}(a,t)$ has the illustration

$$\varphi_{\mathbf{V}}(a,t) = \sum_{k=0}^{n} \varepsilon^{2k}\varphi_{k\mathbf{V}}(a,t) + \varphi_{\mathbf{v}}^n(a,t) \equiv \varphi_{\mathbf{V}}^n(a,t) + \varphi_{\mathbf{v}}^n(a,t), (a,t) \in S$$

Where
$\boldsymbol{\varphi}_{0\mathbf{V}}(a,t) = \varphi(a,t) - \mathbf{U}_0(a,t), \boldsymbol{\varphi}_{k\mathbf{V}}(a,t) = -\mathbf{U}_k(a,t), k \geq 1,$
$\varphi_{\mathbf{v}}^n(a,t) = -\mathbf{v}_{\mathbf{U}}^n(a,t), (a,t) \in S.$

Let the decomposition of the singular part of the boundary value issue solution. As the sum of the functions, we construct the function V (a, t).
$\mathbf{V}(a,t) = \sum_{k=0}^{n} \varepsilon^{2k}\mathbf{V}_k(a,t) + \mathbf{v}_{\mathbf{V}}^n(a,t) \equiv \mathbf{V}^n(a,t) + \mathbf{v}_{\mathbf{V}}^n(a,t), (a,t) \in \bar{G}$ (13)

The functions $V_k(a,t), (a,t) \epsilon \overline{G}$ i.e., Problem solutions are components of the solitary section of the problem solution.
$L_{(4.18)}$ V_0(a, t)=g (a, t, U_0(a, t) + V_0(a, t)) − g (a, t, U_0(a, t)) , (a, t) ∈ G, V_0(a, t)=ϕ_{0V}(a, t), (a, t) ∈ S;
$L_{(4.18)}$ V_k(a, t)=ε^{-2} {$L_{(4.18)}$) − $L_{(4.12)}$} V_{k-1}(a,t) (14)

$$+\varepsilon^{-2k}\left\{\mathbf{g}\left(a,t,\sum_{k_1=0}^{k}\varepsilon^{2k_1}[\mathbf{U}_{k_1}(a,t)+\mathbf{V}_{k_1}(a,t)]\right)\right.$$
$$\left.-\mathbf{g}^e\left(a,t,\sum_{k_1=0}^{k}\varepsilon^{2k_1}\mathbf{U}_{k_1}(a,t)+\sum_{k_1=0}^{k-1}\varepsilon^{2k_1}\mathbf{V}_{k_1}(a,t)\right)\right\}, (a,t) \in G,$$
$$\mathbf{V}_k(a,t) = \varphi_{k\mathbf{V}}(a,t),\ (a,t) \in S,\ k > 0$$

Where

$$\equiv \varepsilon^2\begin{pmatrix}\frac{\partial}{\partial a_1}\left(x_1^1(a,t)\frac{\partial}{\partial a_1}\right) & 0 \\ 0 & \frac{\partial}{\partial a_1}\left(x_1^2(a,t)\frac{\partial}{\partial a_1}\right)\end{pmatrix} - C(a,t) - P(a,t)\frac{\partial}{\partial t}$$

$L_{(4.18)}$

We derive the estimate for the conditions (4.7), (4.15), using a technique identical to the one described

$$\left|\frac{\partial^{k+k_0}}{\partial a_1^{k_1}\partial a_2^{k_2}\partial t^{k_0}}\mathbf{V}(a,t)\right| \leq M(\varepsilon^{-k_1}+\varepsilon^{K-k-2})\exp(-m\varepsilon^{-1}r(a,\Gamma)) \quad (15)$$
$$(a,t) \in \bar{G},\ k+2k_0 \leq K$$

Here r(a, Γ) is the distance from the point ‘a’ to the boundary Γ, and m is an arbitrary constant from the interval (0, m_0), where $m_0 = c_0^{\frac{1}{2}}(1 - m_{(4.3)})^{1/2}$ for $c_0 = c_{0(4.3)}$

Remark. If condition (14) is violated, we pass the problem from the function u(a, t) to the function u ∗ (a, t), u(a, t) = u ∗ (a, t) exp(α t). We select a value that is sufficiently large to satisfy the criteria.

$$\Psi^i(a,t;\alpha) \equiv \alpha p_0 + c^{ii}(a,t) - g_i^i(a,t) \geq c_0,$$
$$m\Psi^i(a,t;\alpha) \geq \left|c^{ij}(a,t)\right| + g_j^i(a,t), \quad (a,t) \in \bar{G},\ i,j = 1,2,\ i \neq j,$$

where $c_0 > 0$, m is a random constant that fulfils the criterion m < 1 and $p_0 = p_{0(2.3)}$. We return to the function u after estimating the function u (a, t) and its components (a, t). It is not difficult to demonstrate that the constants m and M in an estimate of type (15) produced for the function V (a, t) in that case are dependent on α. Furthermore, the constant m = m (α) can be selected arbitrarily small, and the constant M = M (α) rises as α → ∞. As a result, the statement of Theorem is preserved even when condition (4.3b) is violated.

2. Conclusion

Using a parameter-uniform numerical scheme, we have presented the analysis for a class of singularly perturbed linear and semilinear parabolic convection-diffusion problems in this work. To discretize the domain, a uniform mesh has been utilised in the temporal direction and a piecewise-uniform Shishkin mesh in the spatial direction. We introduced the backward-Euler strategy for the time semi-discretization and the upwind difference technique for the spatial semi-discretization.

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