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Dynamic Efficacy of Soil Pollution Using Fractional Order Model-An Adomian Decomposition Approach

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Abstract:- In this paper, a nonlinear fractional order eco-epidemic model of soil pollution is considered with four compartments namely, Susceptible soil (S), Polluted soil (P), Remediate soil (T) and Recovered soil (R). The local and global stability of both pollution free equilibrium and pollution extinction equilibrium points are studied around the equilibrium points. Also, the non-negativity and existence of unique solution of the model is proved using fixed point theorem. Adomain decomposition method is used to find the approximate solution of the proposed model. Numerical simulations of the model are carried out through MATLAB which helps to understand the role of parameters and to validate the theoretical findings.
Keywords*: Soil pollution, fractional order model, Lyapunov function, boundedness and stability, adomain decomposition method.*

1. Introduction

Mathematical model is a tool to provide necessary condition for the dynamical mechanism of spread infectious disease and control the economic loses by using some appropriate strategies. One of the important narrow victories of epidemiology was to formulate, predict and control the disease. Usually in epidemiology the constant population is divided into three compartment namely, Suspended, infectious, and recovered, later on the model extended to various compartment class based on available description. Over the early study the discussion was carried out only in integer order differential equation. Nowday's, the fractional order type of mathematical model plays a widal role in solving many Engineering, epidemiology, ecology and biological model. A fractional calculus considered as the generalization of their order where integer order replaced by fractional order. If the solution of fractional order converges to the solution of the integer order system when the order approaches to one, in this study the integer order system is the special case of fractional order system. The main phenomena are connected with this fractional order system is memory property and hereditary property which are not expressed by any integer order system. Many theoretical description of mechanism fits with frameworks of fractional order system. In the last decades several types of fractional calculus such as Riemann–Liouville, Caputo, Atangana–Baleanu, Caputo–Fabrizio, Katugampola, Hadamard etc. have been introduced to intensively investigate the dynamics of the epidemic models [1-6]. Subtrapaul etl, [16] proposed the dynamical behavior of Covid 19 epidemic SIQR model interms of fractional order. Sung kyochoi [17] etl, studied the h-stability of caputo fractional order differential equation by using some fractional comparison principle and lyapunov direct method. Mukherjee D, Mondal R, deals with fractional order prey-predator model of dynamical system for the reserved area and presents the numerical analysis of hopf type bifurcation. Zizhen Zhang etl [13]., formulate and analyzed the dynamics of new model for COVID 19 epidemics with isolated class of fractional order. Many researchers proposed many model in ecoepidemiology and studied and analyzed the various stability property, existence and nonegativity [8-15]. Numerical approximation of the proposed model analyzed using Rung kutta method, Taylore series method, Adams Bashforth and Adams Moulton method, Adomain and laplaceadomain decomposition method, Homotopy method and generalization of homotopy method [14]. In this work Laplace adomain Decomposition method were used to study the approximation solution of the proposed model. The Laplace adomain decomposition method was introduced by Adomain in 1980. This is very effective method to find the explicit and numerical solution of the physical problem of ordinary and partial differential equation with non-linear initial and boundary condition. It can also be used to study many deterministic, stochastic diffusion methods with and without delay. There is no perturbation and linearization is required and also no extra memory required for solving this problem [12].

The main aim of this paper is to study the mathematical model of caputo fractional order soil pollution. Soil is main requirements for living and non-living organism. In the case of contamination of soil such oil spill, excess uses of heavy metals, inorganic fertilizer then the soils are polluted. All polluted soil contains the compounds which includes metals, inorganic ions and salts (phosphates, carbonates, sulfates, nitrates) and organic compounds (lipids, DNA, fatty acids, alcohol). These compounds are mainly entering through soil microbial activity and decomposition organism (animal and plants) [17-20]. The soil pollution has an unfavorable two-way impact on food security: due to toxic contamination which can reduce the crop yields and unsafe for consumption by human and animals. Currently the degradation of soil and land is affected 3.2 million people that is 40% of the world population. It is necessary government to help and reverse the damage soil and encourage better soil management practices to limit soil pollution.

In this present study the work is carried out as follows: In section 2, the mathematical model of caputo fractional order SPTR is considered. Section 3, discuss the analysis the steady state solution such as existence, uniqueness and non-negativity. Also the two main equilibrium point (pollution free equilibrium and pollution extinction equilibrium) has been found. The local and global stability are studied using basic reproduction number and other condition. In section 4, the numerical approximation to the solution of the proposed model is analyzed. Final section gives the numerical simulation for the validation of parameter.

2. Description of Caputo fractional order SPTR model

Here the dynamical soil pollution model consists of four compartment at any time instant of the real valued fractional order differential equation S(t), P(t), T(t) and R(t) named as susceptible soil area, polluted soil area, under necessary remedial applied to the soil and the recovered subclasses respectively. Soils are usually polluted by some excess of heavy metals and some unwanted use and through plastic things. So if necessary organic and inorganic remedies applied to the soil under treatment (T) class some may recovered but some area of the soil again re-enter to the susceptible soil due to failure of experiment. Nowadays, usage of heavy metals is increasing in day today life span so that recovered soil under treatment is re-enter to the susceptible soil again. Based on the above assumption, the transmission of soil pollution under the treatment measures are shown in the compartment figure (1.1). The models govern the followcing system of equation.

$$\begin{aligned} {}_0^C D^\alpha S(t) &= B - \alpha SP + \beta T + \delta R - \mu S \\ {}_0^C D^\alpha P(t) &= \alpha SP - \gamma P - \alpha_1 P - \mu P \\ {}_0^C D^\alpha T(t) &= \alpha_1 P - \beta T - \gamma_1 T - \mu T \\ {}_0^C D^\alpha R(t) &= \gamma P + \gamma_1 T - \delta R - \mu R \end{aligned} \tag{1.1}$$

with the initial conditions of the above model is, $\begin{cases} S(0) = S^0 \geq 0, \\ P(0) = P^0 \geq 0, \\ T(0) = T^0 \geq 0, \\ R(0) = R^0 \geq 0. \end{cases}$ (1.2)

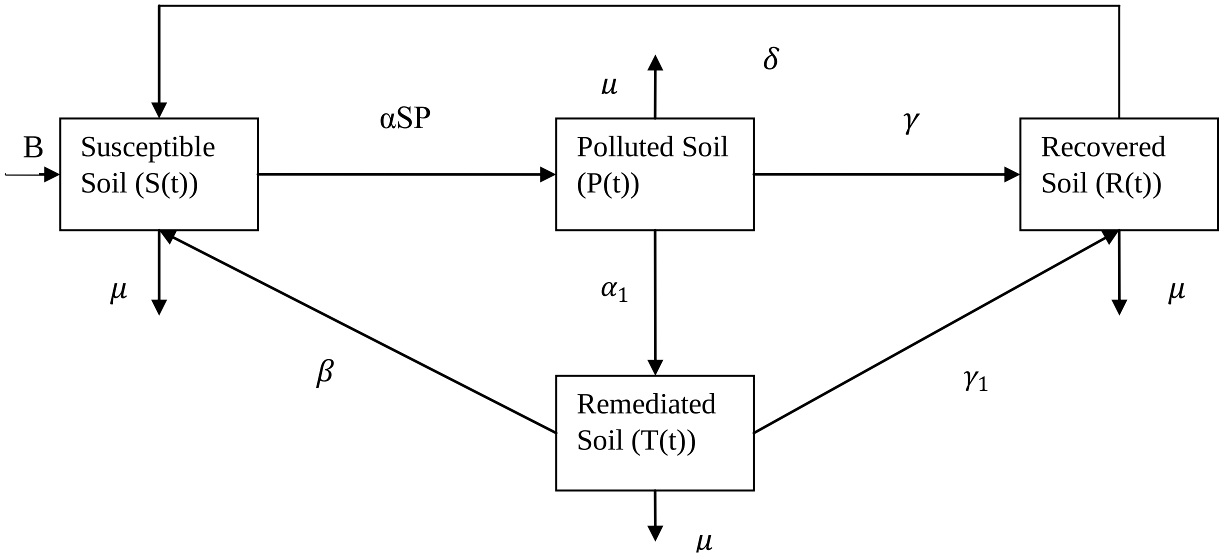

Fig 1: Schematic diagram for the soil pollution

The initial steady states and parameter descriptions are presented in the following table,

Parameter	Description	Value	Source
S(0)	Initial value of the susceptible soil	100	Assumed
P(0)	Initial value of the polluted soil	10	Assumed
T(0)	Initial value of the treated soil	10	Assumed
R(0)	Initial value of the recovered soil	10	Assumed
B	Recruitment rate	365	Estimated
α	Infection rate	0.015	[18]
β	Recovered re-enter to the susceptible rate due to failure of the treatment	0.03	Estimated
γ	Natural recovered rate	0.6	Assumed
α_1	Treatment rate	0.7	Assumed
γ_1	Recovered rate due to treatement	0.2	[18],[19]
μ	Natural death rate	0.025	[20]

Table 1: Details of Parameter values in the given model

3. Analysis of the steady states:

In this section, the mathematical analysis of the proposed model (1.1) is derived.

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Global existence and uniqueness of the solution:

For fractional order system of existence and uniqueness solution are studied in the region $R_M \times (0,t)$, where $R_M = \{(S,P,T,R) \in R^+ : \max(|S|,|P|,|T|,|R| \leq \eta)\}$.

Theorem 3.1 Consider $Y_0 = (S_0,P_0,T_0,R_0) \in R_M$, for each Y_0 there exist a unique solution $Y(t) \in R_M$ of the system (1.1) with initial condition Y_0 for all $t \geq 0$.

Proof:

Let us consider the approach of [9] and take $R(Y) = (R_1(Y), R_2(Y), R_3(Y), R_4(Y))$,

$$
\begin{aligned}
R_1(Y) &= B - \alpha SP + \beta T + \delta R - \mu S \\
R_1(Y) &= \alpha SP - \gamma P - \alpha_1 P - \mu P \\
R_3(Y) &= \alpha_1 P - \beta T - \gamma_1 T - \mu T \\
R_4(Y) &= \gamma P + \gamma_1 T - \delta R - \mu R
\end{aligned} \tag{3.1}
$$

The system of equation (2.1) is reduces to,

$$
\begin{aligned}
R_1(Y) &= B - \alpha SP + \beta T + \delta R - \mu S \\
R_1(Y) &= \alpha SP - AP \\
R_3(Y) &= \alpha_1 P - BT \\
R_4(Y) &= \gamma P + \gamma_1 T - CR
\end{aligned} \tag{3.2}
$$

where, $A = \gamma + \alpha_1 + \mu$, $B = \beta + \gamma_1 + \mu$, $C = \delta + \mu$. For any $Y, \check{Y} \in R_M$ it follows that,

$$
\begin{aligned}
[\![R(Y) - R(\check{Y})]\!] &= |R_1(Y) - R_1(\check{Y})| + |R_2(Y) - R_2(\check{Y})| + |R_3(Y) - R_3(\check{Y})| + |R_4(Y) - R_4(\check{Y})| \\
&= |B - \alpha SP + \beta T + \delta R - \mu S - B + \alpha \check{S}\check{P} + \beta \check{T} + \delta \check{R} - \mu \check{S}| + |\alpha SP - AP - \alpha \check{S}\check{P} + A\check{P}| \\
&\quad + |\alpha_1 P - BT - \alpha_1 \check{P} + B\check{T}| + |\gamma P + \gamma_1 T - CR - \gamma \check{P} - \gamma_1 \check{T} + C\check{R}| \\
&\leq \alpha\zeta|S - \check{S}| + \alpha\zeta|P - \check{P}| + \beta|T - \check{T}| + \delta|R - \check{R}| + \mu|S - \check{S}| + \alpha\zeta|S - \check{S}| + \alpha\zeta|P - \check{P}| + A|S - \check{S}| \\
&\quad + \alpha_1|P - \check{P}| + B|T - \check{T}| + \gamma|P - \check{P}| + \gamma_1|T - \check{T}| + C|R - \check{R}| \\
&\leq (2\alpha\zeta + \mu)|S - \check{S}| + (2(\alpha\zeta + \alpha_1 + \gamma) + \mu)|P - \check{P}| + (2(\beta + \gamma_1) + \mu)|T - \check{T}| + (2\delta + \mu)|R - \check{R}| \\
&\leq H \left\|(S,P,T,R) - (\check{S},\check{P},\check{T},\check{R})\right\| \\
&\leq H\|Y - \check{Y}\|.
\end{aligned}
$$

where $H = \max\{(2\alpha\zeta + \mu), (2(\alpha\zeta + \alpha_1 + \gamma) + \mu),\ 2(\beta + \gamma_1) + \mu,\ 2\delta + \mu\}$

Hence $R(Y)$ satisfies the Lipshitz condition and so existence and uniqueness of fractional order system (1.1) with the initial conditions are established.

Boundedness and Non-negativity

This section studies the interacting area of the Caputo fractional order system is bounded and non -negative at time [0,T).

Theorem 3.2: Prove that the system of fractional order of (1.1) in $R^+$is non-negative and uniformly bounded.

Proof:

The approach used by [9] is followed. Define the function $W(t) = S(t) + P(t) + T(t) + R(t)$ and

$$
{}_0^C D^\alpha W(t) = {}_0^C D^\alpha S(t) + {}_0^C D^\alpha P(t) + {}_0^C D^\alpha T(t) + {}_0^C D^\alpha R(t)
$$

$= B - \alpha SP + \beta T + \delta R - \mu S + \alpha SP - \gamma P - \alpha_1 P - \mu P + \alpha_1 P - \beta T - \gamma_1 T - \mu T + \gamma P + \gamma_1 T - \delta R - \mu R$

$= B - (S + P + T + R)\mu$

$= B - W(t)\mu$

$${}_0^C D^\alpha W(t) + W(t)\mu \leq B$$

Apply the standard comparison theorem [10] for the fractional order model in (1.1) then we have

$$0 \leq W(t) \leq W(0)E_\alpha(-\mu^\alpha) = \frac{B}{\mu} t^\alpha E_{\alpha,\beta}(-t^\alpha),$$

where E_α is the MittagLeffler function. By taking $t \to \infty$, $0 \leq W(t) \leq \frac{B}{\mu}$. Therefore, the solution is uniformly bounded in the region $R = \left\{(S,P,T,R) \in R_+^4, W(t) \leq \frac{B}{\mu} + \epsilon, \epsilon > 0\right\}$. Next, The non - negativity of the solution is studied for the fractional order system (1.1).

Consider (1.1)

${}_0^C D^\alpha S(t)_{S=0} = B, {}_0^C D^\alpha P(t)_{P=0} = 0, {}_0^C D^\alpha T(t)_{T=0} = 0, {}_0^C D^\alpha R(t)_{R=0} = 0$

Hence the solution of the system is non-negative.

Equilibrium Point Analysis:

There are two equilibrium points exists for the model, that are found by equating the time derivatives in system (1.1),

$$B - \alpha SP + \beta T + \delta R - \mu S = 0$$
$$\alpha SP - \gamma P - \alpha_1 P - \mu P = 0$$
$$\alpha_1 P - \beta T - \gamma_1 T - \mu T = 0$$
$$\gamma P + \gamma_1 T - \delta R - \mu R = 0 \qquad (3.3)$$

The two steady states equilibrium point of the proposed model (1.1) is given by,

Pollution free equilibrium point : $P_0\left(\frac{B}{\mu}, 0,0,0\right)$ (3.4)

Pollution extinct equilibrium point : $P_*(S^*, P^*, T^*, R^*)$ (3.5)

where $S^* = \frac{\gamma+\alpha_1+\mu}{\alpha}$, $T^* = \frac{\alpha_1 P^*}{\beta+\gamma_1+\mu}$, $R^* = \frac{1}{\delta+\mu}\left[\gamma + \frac{\gamma_1\alpha_1}{\beta+\gamma_1+\mu}\right]P^*$

$$P^* = \frac{\mu(\gamma + \alpha_1 + \mu)}{\alpha}(R_0 - 1)\left[\gamma + \alpha_1 + \mu - \frac{\beta\alpha_1}{\beta + \gamma_1 + \mu} - \frac{1}{\delta}\left(\frac{1}{\delta + \mu}\left(\gamma + \frac{\gamma_1\alpha_1}{\beta + \gamma_1 + \mu}\right)\right)\right]^{-1}$$

Basic Reproduction Number(BRN)

Basic reproduction number average number of polluted. The next generation matrix (NGM) is use to find the BRN of system (1.1)

Consider, ${}_0^C D^\alpha A(t) = F(Y) - V(Y)$,where $F(Y) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \alpha SP \\ 0 \\ 0 \end{bmatrix}$and

$$V(Y)=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\begin{bmatrix}-(\gamma+\alpha_1+\mu)\\ \gamma P+\gamma_1 T-(\delta+\mu)R\\ \alpha_1 P-(\beta+\gamma_1+\mu)T\end{bmatrix}$$

$$F(Y)=\begin{bmatrix}\frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial R}\\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial T} & \frac{\partial f_2}{\partial R}\\ \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial T} & \frac{\partial f_3}{\partial R}\end{bmatrix}=\begin{bmatrix}\alpha S & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$V(Y)=\begin{bmatrix}\frac{\partial V_1}{\partial P} & \frac{\partial V_1}{\partial T} & \frac{\partial V_1}{\partial R}\\ \frac{\partial V_2}{\partial P} & \frac{\partial V_2}{\partial T} & \frac{\partial V_2}{\partial R}\\ \frac{\partial V_3}{\partial P} & \frac{\partial V_3}{\partial T} & \frac{\partial V_3}{\partial R}\end{bmatrix}=\begin{bmatrix}-(\gamma+\alpha_1+\mu) & 0 & 0\\ \alpha_1 & -(\beta+\gamma_1+\mu) & 0\\ \gamma & \gamma_1 & -(\delta+\mu)\end{bmatrix}$$

$$V^{-1}=\frac{1}{ABC}\begin{bmatrix}BC & 0 & \alpha_1\gamma+B\gamma\\ \alpha_1 C & AC & -A\gamma\\ \alpha_1\gamma_1+B\gamma & 0 & AB\end{bmatrix}$$

Where $A=-(\gamma+\alpha_1+\mu), B=-(\beta+\gamma_1+\mu), C=-(\delta+\mu)$.It follows that spectral radius of the matrix is $\rho(FV^{-1})=\max(\lambda_{1,2,3})$ at pollution free equilibrium point P_0. Therefore, the basic reproduction number,

$$R_0=\frac{\alpha B}{\mu(\gamma+\alpha_1+\mu)} \tag{3.6}$$

Theorem: 3.3: If all the eigen values are negative real part then the pollution free equilibrium point $P_0(S_0,0,0,0)$ of the system (1.1) is locally asymptotically stable.

Proof: From the Matignon condition, we observe that the pollution free equilibrium point is locally asymptotically stable if and only if all the eigenvalues λ_i of the Jacobian $(J(P_0))$ satisfy $|\arg(\lambda_i)|>\frac{\alpha\pi}{2}$. Now consider the Jacobian matrix of the system (1.1) is

$$J[P]=\begin{bmatrix}\frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial P} & \frac{\partial F_1}{\partial T} & \frac{\partial F_1}{\partial R}\\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial P} & \frac{\partial F_2}{\partial T} & \frac{\partial F_2}{\partial R}\\ \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial P} & \frac{\partial F_3}{\partial T} & \frac{\partial F_3}{\partial R}\\ \frac{\partial F_4}{\partial S} & \frac{\partial F_4}{\partial P} & \frac{\partial F_4}{\partial T} & \frac{\partial F_4}{\partial R}\end{bmatrix}=\begin{bmatrix}-\alpha P-\mu & \alpha S & \beta & \delta\\ \alpha & -(\gamma+\alpha_1+\mu) & 0 & 0\\ 0 & \alpha_1 & -(\beta+\gamma_1+\mu) & 0\\ 0 & \gamma & \gamma_1 & -(\delta+\mu)\end{bmatrix} \tag{3.7}$$

$$J[P_0]=\begin{bmatrix}-\mu & \frac{\alpha B}{\mu} & \beta & \delta\\ 0 & -(\gamma+\alpha_1+\mu) & 0 & 0\\ 0 & \alpha_1 & -(\beta+\gamma+\mu) & 0\\ 0 & \gamma & \gamma_1 & -(\delta+\mu)\end{bmatrix}$$

Here are the eigen values, $\lambda_1=-\mu, \lambda_2=-(\delta+\mu), \lambda_{3,4}=\begin{bmatrix}-(\beta+\gamma+\mu) & 0\\ \gamma_1 & -(\delta+\mu)\end{bmatrix}$. Therefore, the eigen values $\lambda_3=-(\beta+\gamma+\mu), \lambda_4=-(\delta+\mu)$. It can be observed that all the eigen values of the Jacobian matrix are negative. Hence by using the Matignon condition we obtain the pollution free equilibrium is locally asymptotically stable.

Theorem: 3.4: If pollution extinction equilibrium point$P_*(S^*,P^*,T^*,R^*)$ are globally asymptotically stable iff $\left[2-\frac{T}{T^*}-\frac{T^*}{T}\right]\le 0,\ and\ \left[2-\frac{R}{R^*}-\frac{R^*}{R}\right]\le 0$

Proof:

Consider the following nonlinear Lyapunov function,

$$W(t)=\left(S-S^*-S^*\ln\left(\frac{S}{S^*}\right)\right)+\left(P-P^*-P^*\ln\left(\frac{P}{P^*}\right)\right)+\left(T-T^*-T^*\ln\left(\frac{T}{T^*}\right)\right)+(R-R^*-R^*\ln(\frac{R}{R^*}))$$

Taking Lyapunov fractional derivative on both sides of the above equation,

$$\begin{aligned}{}_0^CD^\alpha W(t)&=\left(1-\frac{S^*}{S}\right){}_0^CD^\alpha S(t)+\left(1-\frac{P^*}{P}\right){}_0^CD^\alpha P(t)+\left(1-\frac{T^*}{T}\right){}_0^CD^\alpha T(t)+\left(1-\frac{R^*}{R}\right){}_0^CD^\alpha R(t)\\
&=\left(1-\frac{S^*}{S}\right)[B-\alpha SP+\beta T+\delta R-\mu S]+\left(1-\frac{P^*}{P}\right)[\alpha SP-(\gamma+\alpha_1+\mu)P]\\
&\quad+\left(1-\frac{T^*}{T}\right)[\alpha_1P-(\beta+\gamma_1+\mu)T]+\left(1-\frac{R^*}{R}\right)[\gamma P+\gamma_1T-(\delta+\mu)R]\\
&=\left(1-\frac{S^*}{S}\right)[\alpha S^*P^*+\beta T^*+\delta R^*-\mu S^*-\alpha SP+\beta T+\delta R-\mu S]+\left(1-\frac{P^*}{P}\right)[\alpha SP-\alpha S^*P]\\
&\quad+\left(1-\frac{T^*}{T}\right)\left[\alpha_1P-\frac{\alpha_1P^*}{T^*}T\right]+\left(1-\frac{R^*}{R}\right)[\gamma P+\gamma_1T-(\frac{\gamma P^*}{R^*}+\frac{\gamma_1T^*}{R^*})R]\end{aligned}$$

After some simplification we obtain,

$$\begin{aligned}&=-\frac{1}{s}[P^*(S^*-S)^2-\mu(S^*-S)^2+\beta(T^*-T)(S^*-S)+\delta(R^*-R)(S^*-S)]+\alpha_1\left[2-\frac{T}{T^*}-\frac{T^*}{T}\right]\\
&\quad+(\gamma+\gamma_1)\left[2-\frac{R}{R^*}-\frac{R^*}{R}\right].\end{aligned}$$

Finally we conclude that, ${}_0^CD^\alpha W(t)\le 0$, if and only if $\left[2-\frac{T}{T^*}-\frac{T^*}{T}\right]\le 0,\left[2-\frac{R}{R^*}-\frac{R^*}{R}\right]\le 0.$

4. Numerical Approximation:

The numerical approximation solution of a system (1.1) is obtained using Laplace adomain decomposition method,

Applying Laplace transform on both sides of equation (1.1),

$$L\left({}_0^CD^\alpha S(t)\right)=L(B-\alpha SP+\beta T+\delta R-\mu S)$$
$$L\left({}_0^CD^\alpha P(t)\right)=L(\alpha SP-(\gamma+\alpha_1+\mu)P)$$
$$L\left({}_0^CD^\alpha T(t)\right)=L(\alpha_1P-(\beta+\gamma_1+\mu)T)$$
$$L\left({}_0^CD^\alpha R(t)\right)=L(\gamma P+\gamma_1T-(\delta+\mu)R)$$

Substitute the initial condition in the above equation,

$$s^\alpha L[S(t)]-s^{\alpha-1}S(0)=L(B-\alpha SP+\beta T+\delta R-\mu S)$$
$$s^\alpha L[P(t)]-s^{\alpha-1}P(0)=L(\alpha SP-(\gamma+\alpha_1+\mu)P)$$
$$s^\alpha L(T(t))-s^{\alpha-1}T(0)=L(\alpha_1P-(\beta+\gamma_1+\mu)T)$$
$$s^\alpha L(R(t))-s^{\alpha-1}R(0)=L(\gamma P+\gamma_1T-(\delta+\mu)R)$$

Now,

$L[S(t)]=\frac{S(0)}{s}+\frac{1}{s^\alpha}L(B-\alpha SP+\beta T+\delta R-\mu S)$

$L[P(t)]=\frac{P(0)}{s}+\frac{1}{s^\alpha}L(\alpha SP-(\gamma+\alpha_1+\mu)P)$

$L\big(T(t)\big) = \frac{T(0)}{s} + \frac{1}{s^\alpha} L(\alpha_1 P - (\beta + \gamma_1 + \mu)T)$

$L\big(R(t)\big) = \frac{R(0)}{s} + \frac{1}{s^\alpha} L(\gamma P + \gamma_1 T - (\delta + \mu)R)$ (4.1)

Consider the infinite series,

$S(t) = \sum_{k=0}^{\infty} S_k(t),\ P(t) = \sum_{k=0}^{\infty} P_k(t)\ , T(t) = \sum_{k=0}^{\infty} T_k(t),\ R(t) = \sum_{k=0}^{\infty} R_k(t)$ (4.2)

Also consider the decomposition of nonlinear terms as follows,

$$S(t)P(t) = \sum_{k=0}^{\infty} A_k(t) \tag{4.3}$$

where the decomposition polynomial A_k is defined as follows,

$$A_k = \frac{1}{\Gamma(k+1)} \frac{d^k}{d\lambda^k}\left[\sum_{i=0}^{k} \lambda^i S_i(t) \sum_{i=0}^{k} \lambda^i P(t)\right]_{\lambda=0}$$

The first three polynomials,

$$A_0 = S_0(t)P_0(t),$$

$$A_1 = S_0(t)P_1(t) + S_1(t)P_0(t)$$

$$A_2 = 2S_0(t)P_2(t) + 2S_1(t)P_1(t) + 2S_2(t)P_0(t)$$

Substituting all the above equations in equation (4.1)

$L[\sum_{k=0}^{\infty} S_k(t)] = \frac{S(0)}{s} + \frac{1}{s^\alpha} L(B - \alpha \sum_{k=0}^{\infty} A_k(t) + \beta \sum_{k=0}^{\infty} T_k(t) + \delta \sum_{k=0}^{\infty} R_k(t) - \mu \sum_{k=0}^{\infty} S_k(t))$

$L[\sum_{k=0}^{\infty} P_k(t)] = \frac{P(0)}{s} + \frac{1}{s^\alpha} L(\alpha \sum_{k=0}^{\infty} A_k(t) - (\gamma + \alpha_1 + \mu) \sum_{k=0}^{\infty} P_k(t))$

$L(\sum_{k=0}^{\infty} T_k(t)) = \frac{T(0)}{s} + \frac{1}{s^\alpha} L(\alpha_1 \sum_{k=0}^{\infty} P_k(t) - (\beta + \gamma_1 + \mu) \sum_{k=0}^{\infty} T_k(t))$

$L(\sum_{k=0}^{\infty} R_k(t)) = \frac{R(0)}{s} + \frac{1}{s^\alpha} L(\gamma \sum_{k=0}^{\infty} P_k(t) + \gamma_1 \sum_{k=0}^{\infty} T_k(t) - (\delta + \mu) \sum_{k=0}^{\infty} R_k(t))$ (4.4)

Comparing both sides of equations of (4.4), $k \geq 1$

$L(S_0) = \frac{N_1}{s},$

$L(S_1) = \frac{1}{s^\alpha} B - \frac{\alpha}{s^\alpha} L(A_0) + \frac{\beta}{s^\alpha} L(T_0) + \frac{\delta}{s^\alpha} L(R_0) - \frac{\mu}{s^\alpha} L(S_0))$

$L(S_2) = \frac{1}{s^\alpha} B - \frac{\alpha}{s^\alpha} L(A_1) + \frac{\beta}{s^\alpha} L(T_1) + \frac{\delta}{s^\alpha} L(R_1) - \frac{\mu}{s^\alpha} L(S_1))$

$\vdots$

$L(S_{k+1}) = \frac{1}{s^\alpha} B - \frac{\alpha}{s^\alpha} L(A_k) + \frac{\beta}{s^\alpha} L(T_k) + \frac{\delta}{s^\alpha} L(R_k) - \frac{\mu}{s^\alpha} L(S_k))$

$L(P_0) = \frac{N_2}{s},$

$L(P_1) = \frac{\alpha}{s^\alpha} L(A_0) - (\gamma + \alpha_1 + \mu) L(P_0))$

$L(P_2) = \frac{\alpha}{s^\alpha} L(A_1) - \frac{\gamma + \alpha_1 + \mu}{s^\alpha} L(P_1))$

$\vdots$

$L(P_{k+1}) = \frac{\alpha}{s^\alpha} L(A_k) - \frac{\gamma + \alpha_1 + \mu}{s^\alpha} L(P_k))$

$L(T_0) = \frac{N_3}{s},$

$L(T_1) = \frac{1}{s^\alpha}\alpha_1 L(P_0) - \frac{\beta+\gamma_1+\mu}{s^\alpha}L(T_0))$

$L(T_2) = \frac{1}{s^\alpha}\alpha_1 L(P_1) - \frac{\beta+\gamma_1+\mu}{s^\alpha}L(T_1))$

$\vdots$

$L(T_{k+1}) = \frac{1}{s^\alpha}\alpha_1 L(P_k) - \frac{\beta+\gamma_1+\mu}{s^\alpha}L(T_k))$

$L(R_0) = \frac{N_4}{s},$

$L(R_1) = \frac{1}{s^\alpha}\gamma\, L(P_0) + \frac{1}{s^\alpha}\gamma_1\, L(T_0) - (\delta+\mu)L(R_0)$

$L(R_2) = \frac{1}{s^\alpha}\gamma\, L(P_1) + \frac{1}{s^\alpha}\gamma_1\, L(T_1) - (\delta+\mu)L(R_1)$

$\vdots$

$L(R_{k+1}) = \frac{1}{s^\alpha}\gamma\, L(P_k) + \frac{1}{s^\alpha}\gamma_1\, L(T_k) - (\delta+\mu)L(R_k)$

Taking Laplace inverse transform (LIT) on both sides of the above equation,

$S_0 = N_1$

$S_1 = \frac{Bt^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha N_1 N_2 + \beta N_3 + \delta N_4 - \mu N_1]$

$S_2 = \frac{Bt^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha A_1 + \beta T_1 + \delta R_1 - \mu S_1]$

$P_0 = N_2$

$P_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha N_1 N_2 - (\gamma + \alpha_1 + \mu)N_2]$

$P_2 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha A_1 - (\gamma + \alpha_1 + \mu)N_1]$

$T_0 = N_3$

$T_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha_1 N_2 - (\beta + \gamma_1 + \mu)N_3]$

$T_2 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\alpha_1 P_1 - (\beta + \gamma_1 + \mu)T_1]$

$R_0 = N_4$

$R_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\gamma N_2 + \gamma_1 N_3 - (\delta + \mu)N_4]$

$R_2 = \frac{t^\alpha}{\Gamma(\alpha+1)}[\gamma P_1 + \gamma_1 T_1 - (\delta + \mu)R_1]$

Then we have,

$$S(t) = N_1 + S_1 + S_2 + S_3 + \cdots$$
$$P(t) = N_2 + P_1 + P_2 + P_3 + \cdots$$
$$T(t) = N_3 + T_1 + T_2 + T_3 + \cdots$$
$$R(t) = N_4 + R_1 + R_2 + R_3 + \cdots$$

Using the values in the table1, we obtain the required approximation solution for the proposed model.

Numerical Simulation and Discussion:

In this section, the graphical representation of developed model is discussed for the parameter values and initial values given in table 1. For this we use MATLAB (fde12) to plot the graph for the proposed model in (1.1). Here, $h = 2^{-6}$ and fractional order is taken as 0.85 given in figure2. From figure 2 we observe that increase of necessary remediance applied to the polluted soil there must be increase in recovered soil and decrease in suspected soil rate.

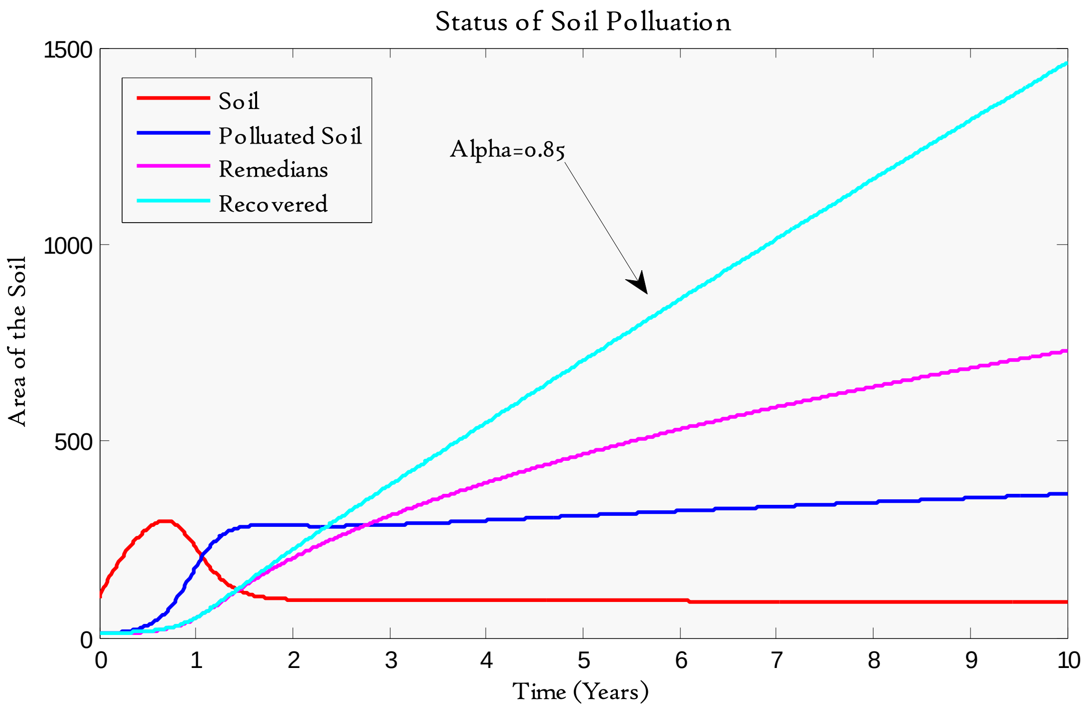

Fig.2: Status of soil pollution in all the compartment at the values of order $\alpha = 0.85$.

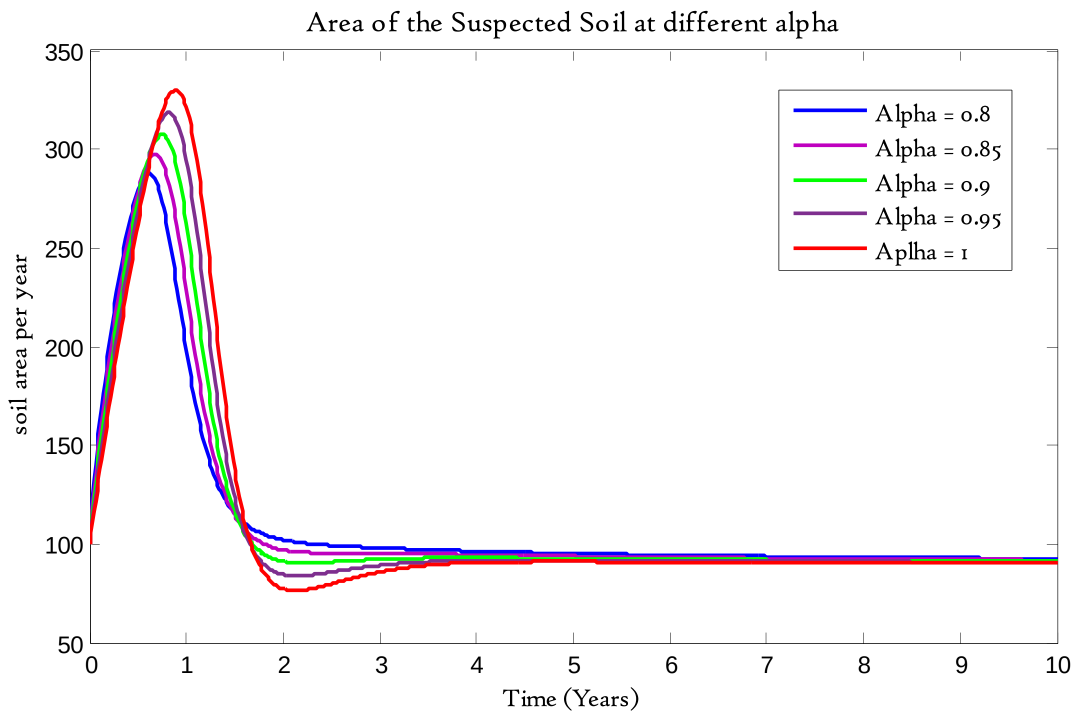

Fig. 3: Total area of the suspected soil at different fractional order α.

Figure 3 is devoted that there is increase in the fractional order of alpha so that the conduct rate from susceptible to infected will be decrease. From fig 4, fig 5, fig 6 we observed that increase in the fractional order decrease in the transition rate from polluted, treated and recovered soil respectively.

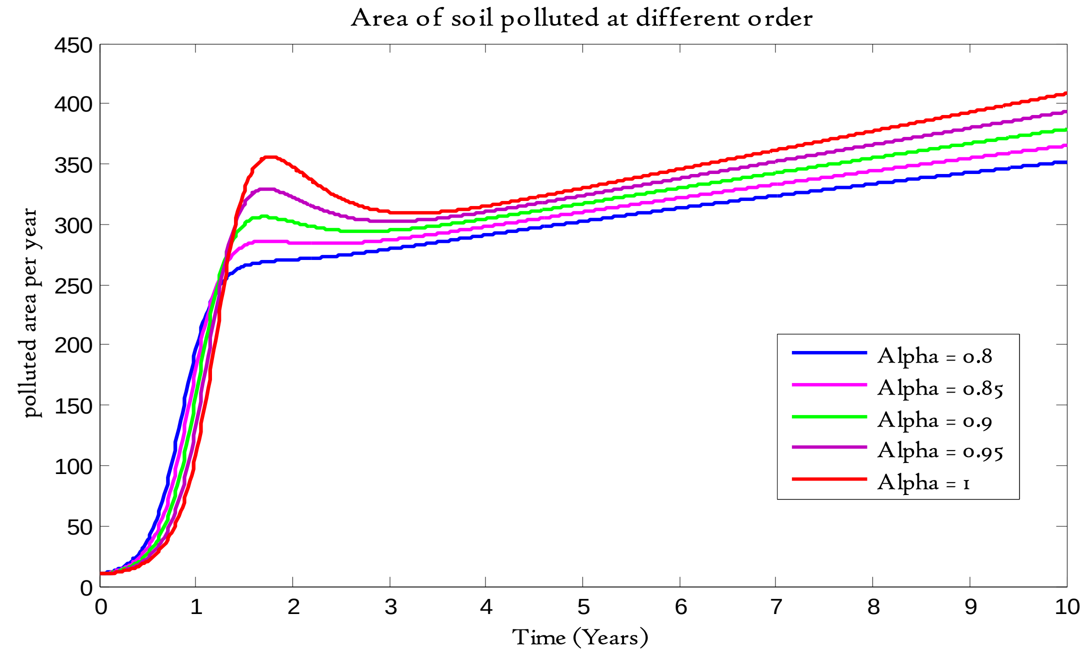

Fig 4: Total area of the polluted soil at different fractional order α.

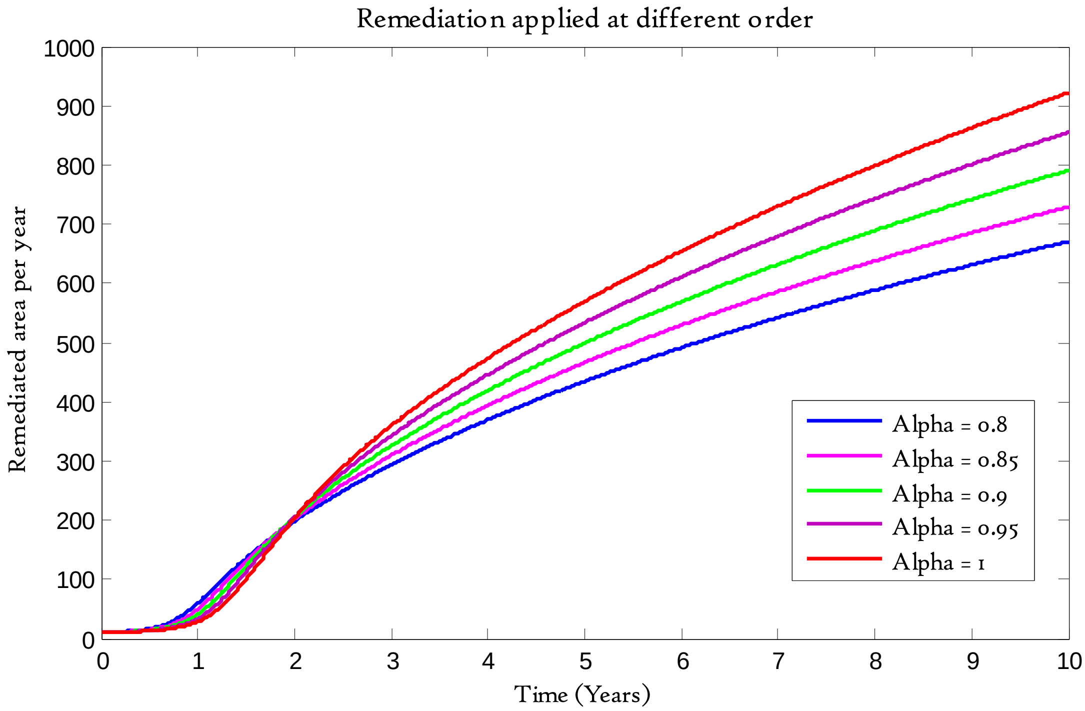

Fig 5: Total area of the soil under treatment with various remediation applied at different fractional order α.

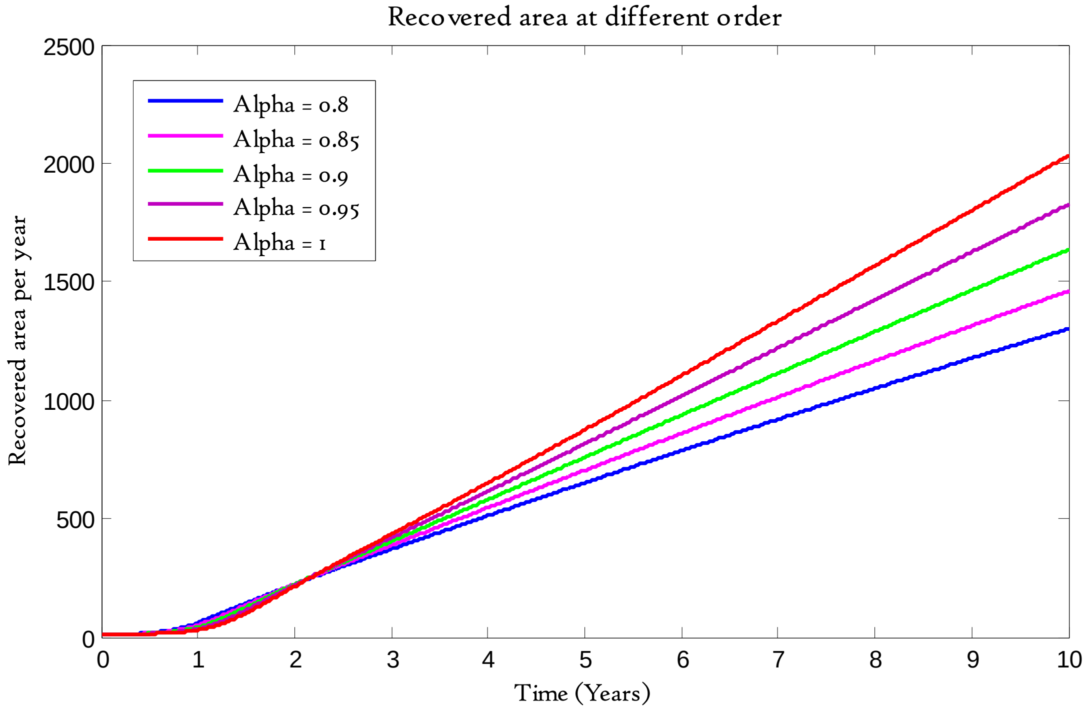

Fig 6: Total area of the soil after treatment those area of the soil recovered different fractional order α.

Conclusion:

In this paper, a nonlinear caputo fractional order eco-epidemic model of soil pollution SPTR model is considered with four compartments namely, Susceptible soil (S), Polluted soil (P), Remediate soil (T) and Recovered soil (R). The non-negative and unique existence of solution of the model is proved using fixed point theorem. The local and global stability of both pollution free equilibrium and pollution extinction equilibrium points are also studied the effectiveness of given model. Laplace Adomain decomposition method is used to find the approximate solution of the proposed model. Numerical results for the assumed parameters describe the behavior of the dynamical system. The achievements of the parameters can be implementing the effective necessary remedies applied to those polluted soil will be recovered by increasing the treatment rate. So, it is very much essential to apply necessary remedies and some precaution to prevent the soil from the adverse pollution.

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