The Monophonic Metric Dimension of Degree Splitting Graph

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Abstract: Let $G = (V, E)$ be a simple graph and $M = \{v_1, v_2, \dots, v_k\} \subset V(G)$ be an ordered set and $v \in V(G)$. The representation $mr(v/M)$ of v with respect to M is the k-tuple $\left(d_m(v, v_1), d_m(v, v_2), \dots, d_m(v, v_k)\right)$. Then M is called a monophonic resolving set if different vertices of G have different representations with respect to M. A monophonic resolving set of minimum number of elements is called a minimum monophonic set for G and its cardinality is known as the monophonic metric dimension of G, represented by $mdim(G)$. In this article, we determined the monophonic metric dimension of degree splitting graph.
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1. Introduction

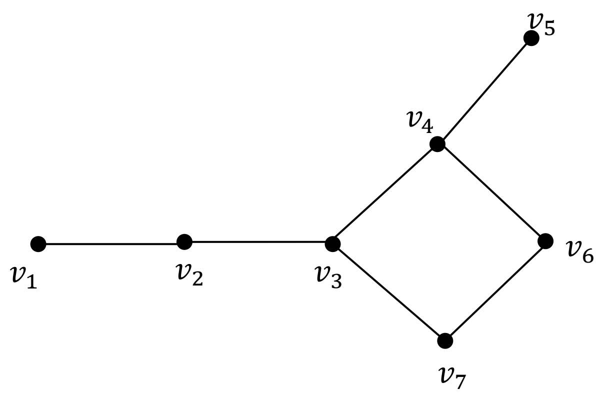
Let $G = (V, E)$ be a simple undirected connected graph. The *order* and *size of* G are denoted by n and m respectively. The length of the shortest $u - v$ path in G is the distance $d(u, v)$ between vertices u and v in a connected graph G. A $u - v$ path with length $d(u, v)$ is referred to as an $u - v$ *geodesic*. For basic graph theoretic terminology, we refer [1]. A path $P's$ chord is an edge that connects two of its non-adjacent vertices. If a path between two vertices u and v in a connected graph G lacks chords, it is referred to as *monophonic path*. The length of the longest $u - v$ monophonic path in G is the monophonic distance $d_m(u, v)$ between $u$and v. These concepts were studied in [3-6].

Let $W = \{w_1, w_2, \dots, w_k\} \subset V(G)$ be an ordered set and $v \in V(G)$. The representation $r(v/W)$ of v with respect to W is the k-tuple $\left(d(v, w_1), d(v, w_2), \dots, d(v, w_k)\right)$. Then W is called a *resolving set* if different vertices of G have different representations with respect to W. A resolving set of minimum number of elements is called a *basis* for G and the cardinality of the basis is known as the *metric dimension* of G, represented by $dim(G)$. These concepts were studied in [2]. In this article, we study a new metric dimension called the *monophonic metric dimension* of a graph. For $M \subseteq V(G)$for each $v \epsilon V$ the monophonic resolving set is $mr\,(v/M) = \left(d_m(v, v_1), d_m(v, v_2) \dots d_m(v, v_k)\right)$, where $M = \{v_1, v_2 \dots . v_k\}$. M is said to be a monophonic resolving set of G, if $mr(v/M) \neq \mathrm{mr}(u/M)$ for every $u, v \in V$, where $u \neq v$. The minimum cardinality of a monophonic resolving set is called the monophonic dimension of G. It is denoted by $mdim(G)$. Any monophonic resolving set of cardinality $mdim(G)$ is called $mdim$-set of G.

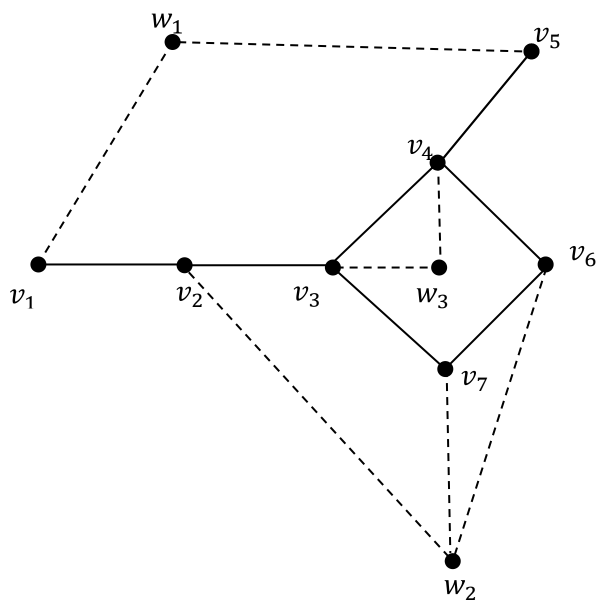
Degree splitting graph: Definition:1.2

Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V - \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \dots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.

Example:1.3 In Figure 1.1, a graph G and the degree splitting graph $DS(G)$ are shown



G



$DS(G)$

Figure 1

Here,$S_1 = \{v_1, v_5\}$,$S_2 = \{v_2, v_7, v_6\}$,$S_3 = \{v_3, v_4\}$,$T = \varphi$.

2. Monophonic Metric Dimension of Degree Splitting Graph

Let us find monophonic metric dimension of degree splitting graph $DS(G)$ of the graphs path, cycle, star, fan, comlete bipartite, wheel, and bistar graph.

Theorem: 2.1 For the Path graph $G = P_n$ $(n \geq 3), mdim\ DS(K_{1,n}) = 2$.

Proof: Let $P_n: v_1, v_2, v_3, \ldots, v_n$ be a path of order n.Since $\deg(v_1) = \deg(v_n) = 1$and $\deg(v_i) = 2; 2 \leq i \leq n-1$, let $S_1 = \{v_2, v_3, v_4, \ldots, v_{n-1}\}$ and $S_2 = \{v_1, v_n\}$ be the two partition of G. To obtain $DS(P_3)$ from P_3 we add u_1, which corresponds to S_2 also P_3 is isomorphic to C_4 and to obtain $DS(P_n)$ for $n \geq 4$ we add a new vertex u_1 and u_2, which corresponds to S_1 and S_2 respectively. As a result, $V(DS(P_3)) = \{u_1, v_1, v_2, v_3\}$ and $V(DS(P_n)) = \{u_1, u_2, v_1, v_2, \ldots, v_n\}$, where $|V(DS(P_n))| = n + 2$ for $n \geq 4$.

Case (i) n is even
Let $M = \{v_2, v_3\}$, $n \geq 4$. Then
$mr(v_1/M) = (1, n-1)$,$mr(v_2/M) = (0,1)$, $mr(v_3/M) = (1,0)$,$mr(v_4/M) = (n-1,1)$,..., $mr(v_{n-1}/M) = (n-2, n-1)$, $mr(v_n/M) = (n-2, n-2)$, $mr(u_1/M) = (n-1, n-2)$, $mr(u_2/M) = (1,1)$.
Since each representations are distinct, M is a monophonic resolving set of G, so that $mdim(DS(P_n)) = 2$**.**

Case(ii) n is odd

Let $M = \{v_2, v_3\}$, $n \geq 4$. Then
$mr(v_1/M) = (1, n-1)$,$mr(v_2/M) = (0,1)$, $mr(v_3/M) = (1,0)$,$mr(v_4/M) = (n-1,1)$, $mr(v_5/M) = (n-2, n-1)$..., $mr(v_{n-1}/M) = (n-3, n-2)$, $mr(v_n/M) = (n-2, n-3)$, $mr(u_1/M) = (n-1, n-2)$, $mr(u_2/M) = (1,1)$.
Since each representations are distinct, M is a monophonic resolving set of G, so that $mdim(DS(P_n)) = 2$**.**

Theorem: 2.2 For the cycle graph $G = C_n$ $(n \geq 3)$, then $mdim\ DS(C_n) = 2$.
Proof : Let $v_1, v_2, \ldots, v_n$ be the cycle C_n. To obtain $mdimDS(C_n)$ for $n \geq 3$. We add a vertex u_1 , which is adjacent to every vertices in C_n. As a result $(DS(C_n)) = \{u_1, v_1, v_2, v_3, \ldots v_n\}$, where $|\ V(DS(C_n))| = n + 1$ for $n \geq 3$. Clearly $DS(C_n)$ is isomorphic to the wheel graph W_n.
Let $M = \{v_1, v_2\}$, we have the following cases,
Case(i) n is even . Then
$mr(u_1/M) = (1,1)$,$mr(v_1/M) = (0,1)$,$mr(v_2/M) = (1,0)$, $mr(v_3/M) = (n-2,1)$,
$mr(v_4/M) = (n-3, n-2)$,$mr(v_5/M) = (n-2, n-3)$, ,..., $mr(v_{n-1}/M) = (n-2, n-3)$, $mr(v_n/M) = (1, n-2)$.
Since each representations are distinct, M is a monophonic resolving set of G, so that $mdim(DS(C_n)) = 2$.
Case (ii) n is odd. Then
$mr(u_1/M) = (1,1)$,$mr(v_1/M) = (0,1)$,$mr(v_2/M) = (1,0)$,
$mr(v_3/M) = (n-2,1)$, $mr(v_4/M) = (n-3, n-2)$,$mr(v_5/M) = (n-3, n-3)$,
$mr(v_6/M) = (n-2, n-3)$,...,$mr(v_{n-1}/M) = (n-2, n-3)$, $mr(v_n/M) = (1, n-2)$.
Since each representations are distinct, M is a monophonic resolving set of G, so that $mdim(DS(C_n)) = 2$**.**

Theorem:2.3 For the complete bipartite graph $G = K_{m,n}$ $(n \geq 3)$.Then $mdimDS(K_{m,n}) = m + n - 2$.
Proof : Consider $K_{m,n}$ with $V(K_{m,n}) = \{u_i, v_j/1 \leq i \leq m, 1 \leq j \leq n\}$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ be two bipartite sets of G. Now we consider the following two cases.
Case (i) $= n$.

In this case each vertex of same degree and so let u_1 be the added vertex, which is adjacent to u_i and v_j, $1 \leq i \leq m$, and $1 \leq j \leq n$. Thus we obtain the graph $DS(K_{m,n})$. Then $DS(K_{m,n}) = \{u_i, v_j/1 \leq i \leq m, 1 \leq j \leq n\}$ and so $DS(K_{m,n}) = m + n + 1$.

Let $M = \{x_1, x_2, x_3, \ldots x_{m-1}, y_1, y_2, y_3, \ldots, y_{n-1}\}$. Then
$mr(u_1/M) = (1,1,1, \ldots ,1,1,1 \ldots ,1)$,$mr(x_1/M) = (0,2,2 \ldots ,2,1,1 \ldots ,1)$,
$mr(x_2/M) = (2,0,2, \ldots ,2,1,1, \ldots ,1)$, $mr(x_3/M) = (2,2,0, \ldots ,2,1,1, \ldots ,1)$,
$mr(x_{m-1}/M) = (2,2, \ldots ,2,0,1,1, \ldots ,1)$,$mr(x_m/M) = (2,2, \ldots ,2,1,1, \ldots ,1)$,

$mr(y_1/M) = (1,1 \dots,1,0,2,2 \dots,2)$,$mr(y_2/M) = (1,1, \dots.,1,2,0,2, \dots,2)$,
$mr(y_3/M) = (1,1, \dots,1,2,2,0, \dots,2)$, … , $mr(y_{n-1}/M) = (1,1, \dots,1,2,2, \dots.,2,0)$,
$mr(y_n/M) = (1,1, \dots,1,2,2, \dots.,2,2)$.

Since each representations are distinct, M is a monophonic resolving set of G. Hence M is a monophonic resolving set of G, so that $mdim\ DS(K_{m,m}) \leq 2m-2$. We prove that $mdim\ DS(K_{m,m}) = 2m-2$.On the contrary, suppose that $mdim\ DS(K_{m,m}) \leq 2m-3$. Then there exist a $mdim$-set M' of $DS(K_{m,m})$ such that $|M'| \leq 2m-3$. Then there exists at least 2 elements $x_i, y_j \in G$, such that $x_i, y_j \notin M'$. Then $mr(x_i/M') = mr(y_j/M')$, Which is a contradiction. Therefore $mdimDS(K_{m,m}) = 2m-2$.

Case (i) $\neq n$.

In this case each vertex u_i is of same degree and each vertex v_j is of same degree where $deg(u_i) \neq$ deg (v_j), $1 \leq i \leq m$, and $1 \leq j \leq n$ so let u_1 and u_2 be the added vertex, where u_1 is adjacent to every u_i and u_2 is adjacent to every v_j. Thus we obtain the graph $DS(K_{m,n})$. Then $DS(K_{m,n}) = \{u_i, v_j, u\_1, u\_2/1 \leq i \leq m, 1 \leq j \leq n\}$ and so $DS(K_{m,n}) = m+n+2$.

Let $M = \{x_1, x_2, x_3, \dots x_{m-1}, y_1, y_2, y_3, \dots, y_{n-1}\}$. Then $(u_1/M) = (1,1,1, \dots,1,1,1 \dots,1)$,
$mr(u_2/M) = (2,2,2, \dots,2,1,1 \dots,1)$, $mr(x_1/M) = (0,2,2 \dots,2,1,1 \dots,1)$,
$mr(x_2/M) = (2,0,2, \dots,2,1,1, \dots.,1)$, …. ,$mr(x_{m-2}/M) = (2,2, \dots,0,2,1,1, \dots,1)$,
$mr(x_{m-1}/M) = (2,2, \dots 2,0,1,1, \dots,1)$, $mr(x_m/M) = (2,2, \dots 2,2,1,1, \dots,1)$,
$mr(y_1/M) = (1,1 \dots,1,0,2,2 \dots,2)$, $mr(y_2/M) = (1,1, \dots.,1,2,0,2, \dots,2)$,
$mr(y_3/M) = (1,1, \dots,1,2,2,0, \dots,2)$, $mr(y_{n-2}/M) = (1,1, \dots,1,2,2, \dots,0,2)$,
$mr(y_{n-1}/M) = (1,1, \dots,1,2,2, \dots.,2,0)$, $mr(y_n/M) = (2,2, \dots 2,2,1,1, \dots,1)$.

Since each representations are distinct, M is a monophonic resolving set of G. Hence M is a monophonic resolving set of G, so that $mdim\ DS(K_{m,n}) \leq m+n-2$. We prove that $mdim\ DS(K_{m,n}) = m+n-2$. On the contrary, suppose that

$mdim\ DS(K_{m,n}) \leq m+n-3$. Then there exist a $mdim$-set M' of $S(K_{m,n})$, such that $|M'| \leq m+n-3$. Then there exists at least 2 elements $y_i, y_j \in G$ such that $y_i, y_j \notin M'$.
Then$(y_i/M') = mr(y_j/M')$, which is a contradiction. Therefore $mdimDS(K_{m,n}) = m+n-2$.

Theorem:2.4 For the star graph $G = K_{1,n}$ $(n \geq 3)$, $mdim\ Ds(K_{1,n}) = 3$.

Proof: Let $v_1, v_2, v_3, \dots, v_{n-1}$ are the end vertices and x is the full vertex of the star $K_{1,n-1}$ and y be the corresponding vertex which is added to obtain the graph $Ds(K_{1,n})$. Then
$V\left(DS(K_{1,n})\right) = \{x, v_1, v_2, v_3, \dots, v_n, y\}$. Clearly$|V\left(DS(K_{1,n})\right)| = n+2$.
Let $M_1 = \{x, u_1\}$. Then
$mr(x/M) = (0,2)$, $mr(u_1/M) = (2,0)$, $mr(v_1/M) = (1,1)$, $mr(v_2/M) = (1,1)$
$mr(v_3/M) = (1,1)$.
Since $mr(v_1/M_1) = mr(v_2/M_1) = (1,1)$, M_1 is not a monophonic resolving set of G.
Let $M_2 = \{x, v_1\}$ .Then
$mr(x/M) = (0,1)$,$mr(u_1/M) = (2,1)$, $mr(v_1/M) = (1,0)$, $mr(v_2/M) = (1,2)$,
$mr(v_3/M) = (1,2)$.
Since $(v_2/M_2) = mr(v_3/M_2) = (1,2)$, M_2 is not a monophonic resolving set of G.
Let $M_3 = \{v_1, v_2\}$.Then
$mr(x/M) = (1,1)$,$mr(u_1/M) = (1,1)$,$mr(v_1/M) = (0,2)$,
$mr(v_2/M) = (2,0)$, $mr(v_3/M) = (2,2)$.
Since $mr(x/M_3) = mr(u_1/M_3) = (1,1)$,
M_3 is not a monophonic resolving set of G. Therefore $mdim(G) \geq 3$
Let $M = \{x, v_1, v_2\}$.Then
$mr(x/M) = (0,1,1)$,$mr(u_1/M) = (2,1,1)$,$mr(v_1/M) = (1,0,2)$,
$mr(v_2/M) = (1,2,0)$, $mr(v_3/M) = (1,2,2)$.

Since each representations are distinct,M is a monophonic resolving set of G, so that $mdimDS(K_{1,n}) = 3$.

Theorem:2.5 For the Wheel graph $G = W_n$ $(n \geq 3)$, $mdim\ DS(W_n) = 3$.

Proof: Let G be a wheel graph with central vertex x and $\{v_1, v_2, v_3, \ldots, v_n\}$ be the degree 3 vertices. To obtain $DS(W_n)$ for ≥ 4 , we add a new vertex u_1. Clearly$|V(DS(W_n))| = n + 2$.

Let $= \{v_1, v_2, x\}$ $n \geq 5$. Then $mr(v_1/M) = (0,1,1)$,$mr(v_2/M) = (1,0,1)$, $mr(v_3/M) = (n - 2,1,1)$, $mr(v_4/M) = (n - 3, n - 2,1)$,…, $mr(v_{n-1}/M) = (n - 1, n - 2,1)$, $mr(v_n/M) = (1, n - 2,0)$, $mr(x/M) = (1,1,0)$, $mr(u_1/M) = (1,1,2)$.

Since each representations are distinct, M is a monophonic resolving set of G. Hence M is a monophonic resolving set of G, so that $mdim\ DS(W_n) \leq 3$.We prove that $mdim\ DS(W_n) = 3$. On the contrary, suppose that $mdim\ DS(W_n) \leq 2$.Then there exist a $mdim$-set M' of $DS(W_n)$ such that $|M'| \leq 2$. Then there exists at least 2 elements $x, u_1 \in G$ such that $x, u_1 \notin M'$.
Then $mr(x/M) = mr(u_1/M)$,which is a contradiction . Therefore $mdimDS(W_n) = 3$.

Theorem:2.6 For the bistar graph $G = B_{r,s}(n \geq 3)$.Then $mdim\left(DS(B_{r,s})\right) = r + s + 1$.

Proof: Consider the bistar graph $B_{r,s}$ with $V(B_{r,s}) = \{u, v, u_i, v_j/1 \leq i \leq r, 1 \leq j \leq s\}$. Here u_i and $v_j$are the vertices adjacent with$u$ and v. $G = DS(B_{r,s})$. Let x, y be the corresponding vertices which are added to obtain $DS(B_{r,s})$. Then $(DS(B_{r,s})) = \{u, v, u_i, v_j, x, y/1 \leq i \leq r, 1 \leq j \leq s\}$, $r \geq 2, s \geq 2$,and so $\left|V\left(DS(B_{r,s})\right)\right| = r + s + 4$.

Let $M = \{x, v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$.Then
$mr(x/M) = (0,4,4, \ldots ,4,4,4, \ldots ,4)$, $mr(y/M) = (3,1,1 \ldots ,1,1,1, \ldots ,1)$
$mr(u/M) = (1,3,3, \ldots ,3,1,1, \ldots ,1)$, $mr(v/M) = (1,1, \ldots ,1,3,3, \ldots ,3)$,
$mr(u_1/M) = (4,3, \ldots ,3,0,2,2, \ldots ,2)$,$mr(u_2/M) = (4,3,3, \ldots ,3,2,0, \ldots ,2)$,
$mr(v_1/M) = (4,0,2 \ldots 2,3,3, \ldots ,3)$,$mr(v_2/M) = (4,2,0 \ldots ,2,3,3, \ldots ,3)$.

Since each representations are distinct, M is a monophonic resolving set of G, so that $mdim(G) \leq r + s + 1$.We prove that $mdim(G) = r + s + 1$. On the contrary, suppose that $mdim(G) \leq r + s$. Then there exis t $mdim$ set-M' of $DS(G)$,such that $|M'| \leq r + s$.Then M' is not a monophonic resolving set, which is contradiction. Therefore $mdim(G) = r + s + 1$.

Theorem: 2.7 Let G be the fan graph $F_n = K_1 + P_{n-1}$ $(n \geq 4)$. Then $mdim(DS(G)) = 3$.

Proof: Let G be the fan graph F_n $(n \geq 5)$. Let $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.Then $(DS(F_n)) = \{x, v_1, v_2, \ldots, v_n, u_1, u_2)$. Clearly,$|V(DS(F_n))| = n + 3$.

Let us assume that $M = \{v_1, v_2, u_1\}$. Then $mr(x/M) = (1, n - 1,2)$,$mr(v_1/M) = (0,1,1)$,$mr(v_2/M) = (1,0, n - 1)$, $mr(v_3/M) = (n - 1,1, n - 1)$, $mr(v_4/M) = (n - 2, n - 1, n - 2)$,…,$mr(v_{n-2}/M) = (n - 2, n - 2, n - 1)$, $mr(v_{n-1}/M) = (n - 1, n - 2,1)$, $mr(u_1/M) = (1, n - 1,0)$, $mr(u_2/M) = (4,1,3)$.

Since each representations are distinct, M is a monophonic resolving set of G. Hence M is a monophonic resolving set of G, so that $mdim\ DS(F_n) \leq 3$. We prove that $mdim\ DS(F_n) = 3$. On the contrary, suppose that $mdim\ DS(F_n) \leq 2$. Then there exist a $mdim$-set M' of $DS(F_n)$ such that $|M'| \leq 2$. Then there exists at least 2 elements $x, u_1 \in G$ such that $x, u_1 \notin M'$.Then $r(x/M) = mr(u_1/M)$, which is a contradiction . Therefore $mdimDS(F_n) = 3$.

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