Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 4 (2023)

Degree of approximation of signals (functions) by *(C, 2)(E, l)* product means of conjugate series of Fourier series

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Abstract
In this paper, we establish a new theorem to find the Degree of Approximation of signals (Functions) $\chi \in \left(L^r, \zeta(t)\right)$ class by new $(C,\ 2)(E,\ \ell)$ product summability method of conjugate series of Fourier series.

Keywords: *Signals (functions), (C, 2) means,* $(E,\ \ell)$ *means, Lipschitz class and* $(C,\ 2)(E,\ \ell)$ *product summability method, Lip(ζ(t), r) class,* $W\left\{L^r,\ \zeta(t)\right\}$ *class, Fourier Series and conjugate series of Fourier series.*

1. Introduction

It became very interesting to find the estimation of errors of functions by using various product summability means. Now a days people are working in the direction of error estimation of functions belongignt to different spaces by using various summability methods. Rhoads [4], Leindler [18], Sahney and Goel [9] and many more have estimated the error of functions belonging to Lipschitz class and other classes by Cesàro, Nörlund and Euler methods. later on Nigam and Sharma ([2], [15]), recently Kushwaha et al. ([3], [5], [19]), and various investigators ([20]-[27]) have estimated interesting results on degree of approximation of functions by Nörlund-Euler, (C, 2)(E, 1) and Euler-Matrix product summability means of Fourier series and conjugate series of Fourier series respectively. Till now no work seems to have been done so for to find the estiomation of signals (functions) by using $(C,2)(E,\ell)$ summability means of conjugate series of Fourier series. In tis paper we have used second order Cesàro means along with Euler means to obtain the product mean $(C,2)(E,\ell)$ which is very new in present days.

2. Definitions and Notations

Let $\sum_{\nu=0}^{\infty} u_\nu$ be a given infinite series with $\{s_\nu\}$ for it's ν^{th} partial sum.

Let $\{\Omega_\eta^{E_\ell}\}$ denote the sequence of $(E,\ell)=E_\eta^\ell$ means of the sequence $\{s_\nu\}$. If the (E,ℓ) transform of $\{s_\nu\}$ is defined as

$$\Omega_\eta^{E_\ell}(\zeta;x)=\frac{1}{(1+\ell)^\eta}\sum_{\nu=0}^{\eta}\binom{\eta}{\nu}\ell^{\eta-\nu}s_\nu \to s \text{ as } \eta\to\infty \tag{2.1}$$

Then the series $\sum_{\nu=0}^{\infty}u_\nu$ is said to be summable to the number s by (E,ℓ) method. (Hardy [1])

Let $\{\Omega_\eta^{C_2}\}$ denote the sequence of $(C,2)=C_\eta^2$ means of the sequence $\{s_\nu\}$ .If the $(C,2)$ transform of $\{s_\nu\}$ is defined as

$$\Omega_\eta^{C_2}(\zeta;x)=\frac{2}{(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)s_\nu(\zeta;x)\to s \text{ as } \eta\to\infty \tag{2. 2}$$

Then the series $\sum_{\nu=0}^{\infty}u_\nu$ is said to be summable to the number s by the $(C,2)$ method. (Cesàro)

Thus if $(C,2)$ transform of (E,ℓ) transform defines $(C,2)(E,\ell)$ transformation and denoted by $C_\eta^2.E_\eta^\ell$.
Then if

$$\Omega_\eta^{C_2E_\ell}(\zeta;x)=\frac{2}{(\eta+1)(\eta+2)}\left[\sum_{\nu=0}^{\eta}(\eta-\nu+1)\left\{\frac{1}{(1+\ell)^\nu}\sum_{k=0}^{\nu}\binom{\nu}{k}\ell^{\nu-k}\right\}\right]s_k\to s \text{ as } \eta\to\infty \tag{2.3}$$

where $\Omega_\eta^{C_2E_\ell}$ denotes the sequence of C_2E_ℓ means that is $(C,2)(E,\ell)$ product means of the sequence $\{s_\nu\}$.

Then the series $\sum_{\nu=0}^{\infty}u_\nu$ is said to be summable to the number 's' by the $(C,2)(E,\ell)$ method.

We know that $(C,2)(E,\ell)$ method is regular. Let χ be 2π -periodic, lebesgue integrable function on $[-\pi,\ \pi]$ then its Fourier series associated with a point x is defined by

$$\chi(x)=\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)=\sum_{\nu=0}^{\infty}A_\nu,\quad \nu\in N \tag{2.4}$$

and the series

$$\sum_{\nu=1}^{\infty}(a_\nu\sin\nu x-b_\nu\cos\nu x)=-\sum_{\nu=0}^{\infty}T_\nu(x) \tag{2.5}$$

is called the conjugate series of the Fourier series with ν^{th} partial sum $\tilde{\tilde{s}}_\nu(\chi;\ x)$.

We use following notations through out the paper

$$\chi_x(t) = \chi(x+t) + \chi(x-t)$$

$$\tilde{\chi}(x) = -\frac{1}{2\pi}\int_0^{\pi} \chi_x(t)\cot(t/2)\,dt$$

$$\tilde{\kappa}_\eta(t) = \frac{2}{\pi(\eta+1)(\eta+2)}\left[\sum_{\nu=0}^{\eta}(\eta-\nu+1)\left\{\frac{1}{(1+\ell)^{\nu}}\sum_{k=0}^{\nu}\binom{\nu}{k}\ell^{\nu-k}\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right] \tag{2.6}$$

and L_r-norm is defined by

$$\|\chi\|_r = \left(\int_0^{2\pi}|\chi(x)|^r\,dx\right)^{1/r}, \quad r \ge 1$$

and the estimation of errors which is known as degree of approximation of a function ζ given by Zygmund [17]

$$E_\eta(\chi) = \min\|\Delta_\eta(x) - \chi(x)\|_r$$

where $\Delta_\eta(x)$ is some η^{th} degree trigonometric polynomial. This method of approximation is called the trigonometric Fourier approximation.

A function $\chi \in Lip\,\alpha$ if

$$|\chi(x+t) - \chi(x)| = O(|t|^{\alpha}) \text{ for } 0 < \alpha \le 1,\ t > 0.$$

and the function $\chi \in Lip\,(\alpha, r)$ if

$$\left(\int_0^{2\pi}|\chi(x+t) - \chi(x)|^r\,dx\right)^{1/r} = O(|t|^{\alpha}) \text{ for } 0 < \alpha \le 1,\ t > 0.$$

Given a positive increasing function $\zeta(t)$ and an integer $r \ge 1$, $\chi \in Lip\{\zeta(t),\ r\}$ if

$$\left(\int_0^{2\pi}|\chi(x+t) - \chi(x)|^r\,dx\right)^{1/r} = O\{\zeta(t)\}.$$

If $\zeta(t) = t^{\alpha}$ then $Lip\{\zeta(t),\ r\}$ class coincides with the $Lip\,(\alpha, r)$ class and if $r \to \infty$ then $Lip\,(\alpha, r)$ class reduces to $Lip\,\alpha$ class.

A function $\chi \in W\{L^r,\ \zeta(t)\}$ if

$$\left(\int_0^{2\pi}|\chi(x+t) - \chi(x)|^r \sin^{\beta r}(t/2)\,dx\right)^{1/r} = O\{\zeta(t)\},\ \beta \ge 0,\ r \ge 1,\ t > 0.$$

where $\chi(t)$ is increasing function of t.

We observe that

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 44 No. 4 (2023)

$$Lip\ \alpha \subseteq Lip(\alpha,\ r) \subseteq Lip(\zeta(t), r) \subseteq W(L^r, \zeta(t)) \quad \text{for} \quad 0 < \alpha \le 1,\ r \ge 1.$$

Kushwaha [5] has proved a theorem on approximation of function by (C, 2)(E, 1) product summability method as following-

Theorem:- If $f : R \to R$ is 2π periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to $Lip(\alpha, r)$ class then the error estimation of function (signals) by the (C, 2)(E, 1) product means of Fourier series f satisfies

$$t_n^{C_2E_1}(f;\ x) = \frac{2}{(n+1)(n+2)}\left[\sum_{k=0}^{n}(n-k+1)\left\{\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu(f;\ x)\right\}\right] \tag{2.7}$$

of its Fourier series is given by

$$\left\|C_n^2E_n^1 - f\right\|_r = O\left\{\xi\left(\frac{1}{n+1}\right)\right\} \tag{2.8}$$

is the (C, 2)(E,1) means of the Fourier series (2.4).

3. Main Theorem

If a function χ be a 2π periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to $W\{L^r,\ \zeta(t)\}$ class then the estimate error of signals (functions) χ by the $(C, 2)(E, \ell)$ product means of conjugate series of Fourier series of χ satisfies

$$\left\|\tilde{\tilde{\Omega}}_\eta^{(C,\ 2)(E,\ \ell)} - \tilde{\tilde{\chi}}(x)\right\|_r = O\left[\eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right] \tag{3.1}$$

provided that $\zeta(t)$ satisfies following conditions given below-

$$\left\{\int_0^{1/\eta}\left(\frac{t\left|\chi_x(t)\sin^\beta t\right|^r}{\zeta(t)}\right)dt\right\}^{1/r} = O\left(\frac{1}{\eta}\right) \tag{3.2}$$

and

$$\left\{\int_{1/\eta}^{\pi}\left(\frac{t^{-i}\left|\chi_x(t)\right|^r}{\zeta(t)}\right)dt\right\}^{1/r} = O\left(\eta^\varsigma\right) \tag{3.3}$$

where ς is an arbitrary positive number such that $s(\beta-\varsigma)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, $1 \le r < \infty$. These conditions (3.2) and (3.3) hold in $C_\eta^2E_\eta^\ell$ that is $(C, 2)(E, \ell)$ means of the conjugate series of Fourier series.

4.Lemmas

To prove theorem, we need the following lemma:

$$\left|\tilde{\tilde{\kappa}}_n(t)\right| = O\left(\frac{1}{t}\right); \quad \text{for} \quad \frac{1}{\eta} \le t \le \pi\ .$$

Proof For $\frac{1}{\eta} \le t \le \pi$; by applying Jordan's lemma $\sin(t/2) \ge t/\pi$ and $\sin nt \le 1$. .

$$\left|\tilde{\tilde{\kappa}}_n(t)\right| = \frac{1}{\pi(\eta+1)(\eta+2)}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\left[\frac{1}{(1+\ell)^{\nu}}\sum_{k=0}^{\nu}\left\{\binom{\nu}{k}\ell^{\nu-k}\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right]\right|$$

$$= \frac{1}{\pi(\eta+1)(\eta+2)}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\frac{1}{(1+\ell)^{\nu}}\sum_{k=0}^{\nu}\left\{\binom{\nu}{k}\ell^{\nu-k}\frac{\mathrm{Re}\{e^{i(k+1/2)t}\}}{t/\pi}\right\}\right|$$

$$\le \frac{1}{\pi(\eta+1)(\eta+2)}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\frac{1}{(1+\ell)^{\nu}}\sum_{k=0}^{\nu}\left\{\binom{\nu}{k}\ell^{\nu-k}\frac{\{e^{i(k+1/2)t}\}}{t/\pi}\right\}\right|$$

$$= \frac{\pi}{\pi(\eta+1)(\eta+2)t}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\frac{1}{(1+\ell)^{\nu}}\sum_{k=0}^{\nu}\left\{\binom{\nu}{k}\ell^{\nu-k}e^{i(k+1/2)t}\right\}\right|$$

$$= \frac{1}{(\eta+1)(\eta+2)t}\left[\sum_{\nu=0}^{n}(\eta-\nu+1)\left\{\frac{\left|\ell+e^{it}\right|}{(1+\ell)^{k}}\right\}\right]$$

$$= \frac{1}{(\eta+1)(\eta+2)t}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\frac{(1+\ell^2+2\ell\cos t)^{k/2}}{(1+\ell)^{k}}\right|$$

$$= \frac{1}{(\eta+1)(\eta+2)t}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\left\{1-\frac{4\ell\sin^2 t/2}{(1+\ell)^{k}}\right\}\right|$$

$$\le \frac{1}{(\eta+1)(\eta+2)t}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\left\{1-\frac{4\ell\, t^2}{\pi^2(1+\ell)^{k}}\right\}\right| \qquad \because \sin t/2 \ge t/\pi.$$

$$\le \frac{1}{(\eta+1)(\eta+2)t}\left|\sum_{\nu=0}^{n}(\eta-\nu+1)\left\{e^{-\frac{4\ell\, t^2}{\pi^2(1+\ell)^{k}}}\right\}\right|$$

$$\because e^{x}(1-x) < 1 \text{ when } 0 < x < 1$$

$$= \frac{1}{(\eta+1)(\eta+2)t}\left[\sum_{\nu=0}^{n}(\eta-\nu+1)\right]$$

$$= \frac{1}{(\eta+1)(\eta+2)t}\left[\sum_{\nu=0}^{n}(\eta+1)-\sum_{\nu=0}^{n}\nu\right]$$

$$= \frac{\eta(\eta+1)}{(\eta+1)(\eta+2)t} - \frac{\eta(\eta+1)/2}{(\eta+1)(\eta+2)t}$$

$$\cong O\left(\frac{1}{t}\right).$$

5. Proof of the Theorem

Following Titchmarsh [6] and using Riemann-Lebesgue theorem $\tilde{\tilde{s}}_\eta(\chi;x)$ η^{th} partial sum of the Fourier series is given by

$$\tilde{\tilde{s}}_\eta(x)=\frac{-1}{2\pi}\int_0^\pi \cot t/2\ \chi_x(t)\,dt+\frac{1}{2\pi}\int_0^\pi \frac{\cos(\nu+1/2)t}{\sin t/2}\chi_x(t)\ dt$$

$$\tilde{\tilde{s}}_\eta(x)+\frac{1}{2\pi}\int_0^\pi \cot t/2\ \chi_x(t)\,dt=\frac{1}{2\pi}\int_0^\pi \frac{\cos(\nu+1/2)t}{\sin t/2}\chi_x(t)\ dt$$

$$\tilde{\tilde{s}}_\eta(x)+\frac{1}{2\pi}\int_0^{1/\eta} \cot t/2\ \chi_x(t)\,dt+\frac{1}{2\pi}\int_{1/\eta}^{\pi} \cot t/2\ \chi_x(t)\,dt$$

$$=\frac{1}{2\pi}\left(\int_0^{1/\eta}+\int_{1/\eta}^{\pi}\right)\frac{\cos(\nu+1/2)t}{\sin t/2}\chi_x(t)\ dt$$

$$\tilde{\tilde{s}}_\eta(\chi;x)-\tilde{\tilde{\chi}}(x)=\frac{1}{2\pi}\int_0^{1/\eta}\chi_x(t)\left\{\frac{\cos(\nu+1/2)t}{\sin t/2}-\cot\ t/2\right\}dt+\frac{1}{2\pi}\int_{1/\eta}^{\pi}\chi_x(t)\frac{\cos(\nu+1/2)t}{\sin t/2}dt$$

$$=\frac{1}{2\pi}\int_0^{1/\eta}\chi_x(t)\left\{\frac{\cos(\nu+1/2)t-\cos t/2}{\sin t/2}\right\}dt+\frac{1}{2\pi}\int_{1/\eta}^{\pi}\chi_x(t)\frac{\cos(\nu+1/2)t}{\sin t/2}dt$$

$$=\frac{1}{2\pi}\int_0^{1/\eta}\chi_x(t)\left\{\frac{2\sin(\nu+1)t/2\ \ \sin(-\nu t)/2}{\sin t/2}\right\}dt+\frac{1}{2\pi}\int_{1/\eta}^{\pi}\chi_x(t)\frac{\cos(\nu+1/2)t}{\sin t/2}dt$$

$$=\frac{1}{2\pi}\int_0^{1/\eta}\chi_x(t)\{(\nu+1)\sin(-\nu t)/2\}dt+\frac{1}{2\pi}\int_{1/\eta}^{\pi}\chi_x(t)\frac{\cos(\nu+1/2)t}{\sin t/2}dt$$

Using the (E,ℓ) transform of $\tilde{\tilde{s}}_\eta(\chi;x)$ is given by

$$\left|\tilde{\tilde{\Omega}}_\eta^{(E,\ell)}-\tilde{\tilde{\chi}}(x)\right|=\frac{1}{2\pi}\int_0^\pi\chi_x(t)\left[\frac{1}{(1+\ell)^\eta}\sum_{k=0}^{\eta}\binom{\eta}{k}\ell^{\eta-k}\left\{(k+1)\left|\sin(-kt/2)\right|+\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right]dt$$

$$=\frac{1}{2\pi}\int_0^\pi\chi_x(t)\left[\frac{1}{(1+\ell)^\eta}\sum_{k=0}^{\eta}\binom{\eta}{k}\ell^{\eta-k}\left\{(k+1)+\frac{\cos(k+1/2)t}{\sin t/2}\right\}\right]dt$$

Now, denoting $(C,2)(E,\ell)$ transformation of $\tilde{\tilde{s}}_\eta(\chi;x)$ is given by

$$\tilde{\tilde{\Omega}}_{\eta}^{(E,\ell)} - \tilde{\tilde{\chi}}(x) = \frac{1}{\pi(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)$$

$$\times\left[\frac{1}{(1+\ell)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ell^{k-\nu}\int_0^{\pi}\chi_x(t)\left((\nu+1)+\frac{\cos(\nu+1/2)t}{\sin t/2}\right)\right]dt$$

$$= \frac{1}{\pi(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)\left[\frac{1}{(1+\ell)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ell^{k-\nu}\int_0^{1/\eta}\chi_x(t)\left((\nu+1)\right)\right]$$

$$+\frac{1}{\pi(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)\left[\frac{1}{(1+\ell)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ell^{k-\nu}\int_{1/\eta}^{\pi}\chi_x(t)\left(\frac{\cos(\nu+1/2)t}{\sin t/2}\right)\right]$$

$$= \frac{1}{\pi(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)(\nu+1)\left\{\int_0^{1/\eta}\chi_x(t)\right\}$$

$$+\frac{1}{\pi(\eta+1)(\eta+2)}\sum_{\nu=0}^{\eta}(\eta-\nu+1)\left[\frac{1}{(1+\ell)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\ell^{k-\nu}\int_{1/\eta}^{\pi}\chi_x(t)\left(\frac{\cos(\nu+1/2)t}{\sin t/2}\right)\right]$$

$$= O(\eta)\int_0^{1/\eta}\chi_x(t)\,dt + \int_0^{1/\eta}\chi_x(t)\,\tilde{\tilde{\kappa}}_{\eta}(t)dt \qquad \text{(by Lemma)}$$

$$= I_1 + I_2$$

Now, $\quad |I_1| \le \int_0^{1/\eta}|\chi_x(t)||O(\eta)|\,dt$

Further $\chi \in W\{L^r, \zeta(t)\} \Rightarrow \chi_x \in W\{L^r, \zeta(t)\}$, thus

$$|I_1| \le \int_0^{1/\eta}\left|\frac{t\ \chi_x(t)\ \sin^{\beta} t}{\zeta(t)}\cdot\frac{\zeta(t)\ O(\eta)}{t\sin^{\beta} t}\right|dt$$

Using Hölder's Inequality and the fact that $\chi_x(t) \in W\{L^r, \zeta(t)\}$ and using the lemma-

$$|I_1| \le \left(\int_0^{1/\eta}\left|\frac{t\ \chi_x(t)\ \sin^{\beta} t}{\zeta(t)}\right|^r dt\right)^{1/r}\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\eta}\left|\frac{\zeta(t)\ O(\eta)}{t\sin^{\beta} t}\right|^s dt\right)^{1/s}$$

$$= O\left(\frac{1}{\eta}\right)\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\eta}\left|\frac{\zeta(t)\ O(\eta)}{t\sin^{\beta} t}\right|^s dt\right)^{1/s} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1$$

$$= \left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\eta}\left(\frac{\zeta(t)}{t\sin^{\beta} t}\right)^s dt\right)^{1/s}$$

Since $\zeta(t)$ is a positive increasing function and using second mean value theorem for integrals-

$$I_1 = \zeta\left(\frac{1}{\eta}\right)\left\{\int_{\varepsilon}^{1/\eta} \frac{1}{t^{(\beta+1)s}}\,dt\right\}^{1/s}$$

$$= \zeta\left(\frac{1}{\eta}\right)\left[\left\{\frac{t^{-(\beta+1)s+1}}{-(\beta+1)s+1}\right\}_{\varepsilon}^{1/\eta}\right]^{1/s} \quad \text{for some } 0 \le \varepsilon \le 1/\eta$$

$$= O\left\{\eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right\} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.$$

Now consider,

$$|I_2| \le \int_{1/\eta}^{\pi} |\chi_x(t)|\left|\tilde{\tilde{\kappa}}_\eta(t)\right| dt$$

Using Hölder's inequality

$$|I_2| \le \left[\int_{1/\eta}^{\pi}\left\{\frac{t^{-\varsigma}\,|\chi_x(t)|\,\sin^{\beta} t}{\zeta(t)}\right\}^r dt\right]^{1/r}\left[\int_{1/\eta}^{\pi}\left\{\frac{\zeta(t)\left|\tilde{\tilde{\kappa}}_\eta(t)\right|}{t^{-\varsigma}\sin^{\beta} t}\right\}^s dt\right]^{1/s}$$

$$= O\{(\eta)^{\varsigma}\}\left[\int_{1/\eta}^{\pi}\left\{\frac{\zeta(t)\left|\tilde{\tilde{\kappa}}_\eta(t)\right|}{t^{-\varsigma+\beta}}\right\}^s dt\right]^{1/s}$$

$$= O\{(\eta)^{\varsigma}\}\left[\int_{1/\eta}^{\pi}\left\{\frac{\zeta(t)}{t^{1-\varsigma+\beta}}\right\}^s dt\right]^{1/s} \quad \text{by lemma}$$

Now putting $t = \frac{1}{x}$ and $dt = -\frac{1}{x^2}dx$

$$|I_2| = O\{\eta^{\varsigma}\}\left[\int_{1/\pi}^{\eta}\left\{\frac{\zeta(1/x)}{x^{\varsigma-1-\beta}}\right\}^s \frac{dx}{x^2}\right]^{1/s}$$

Since $\zeta(t)$ is a positive increasing function and using second mean value theorem for integrals-

$$|I_2| = O\left\{\eta^{\varsigma}\,\zeta\left(\frac{1}{\eta}\right)\right\}\left[\int_{\tau}^{\eta}\frac{dx}{x^{s(\varsigma-1-\beta)+2}}\right]^{1/s}, \text{ for some } 1/\pi \le \tau \le \eta$$

$$= O\left\{\eta^{\varsigma}\ \zeta\left(\frac{1}{\eta}\right)\right\}\left[\left\{\frac{x^{\,s(1-\varsigma+\beta)-1}}{s(1-\varsigma-\beta)-1}\right\}_{\tau}^{\eta}\right]^{1/s}$$

$$= O\left\{\eta^{\varsigma}\ \zeta\left(\frac{1}{\eta}\right)\right\}\eta^{(1-\varsigma+\beta)-1/s}$$

$$= O\left\{\ \zeta\left(\frac{1}{\eta}\right)\right\}\eta^{\beta+1-1/s}$$

$$= O\left\{\ \eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right\},\quad \text{since } \frac{1}{r}+\frac{1}{s}=1.$$

Combining I_1 and I_2 yields-

$$\left|\tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x)\right| = O\left[\eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right]$$

Now, using L_r-norm, we get

$$\left\|\tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x)\right\|_r = \left\{\int_0^{2\pi}\left|\tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x)\right|^r dx\right\}^{1/r}$$

$$= O\left[\int_0^{2\pi}\left\{\eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right\}^r dx\right]^{1/r}$$

$$= O\left\{\eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right\}\left\{\left(\int_0^{2\pi} dx\right)^{1/r}\right\}$$

$$= O\left\{\ \eta^{\beta+1/r}\zeta\left(\frac{1}{\eta}\right)\right\}.$$

6. Particular Cases Some

1. If $\beta = 0$ then $W\{L^r,\ \zeta(t)\}$ class reduces to $Lip\{\zeta(t), r\}$ class $r \geq 1$, then the estimation of error of the signals (functions) by $(C,2)(E,\ell)$ means is given by

$$\left\|\tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x)\right\| = O\left\{\ \eta^{1/r}\zeta\left(\frac{1}{\eta}\right)\right\},\ \ 1/r < \alpha < 1. \qquad (6.1)$$

2. If $\zeta(t) = t^{\alpha}, 0 \leq \alpha < 1$ in case (1) then $Lip\{\zeta(t), r\},\ r \geq 1$ class reduces to $Lip(\alpha,\ r)$ class, then the estimation of the error of signals (functions) by $(C,2)(E,\ell)$ means is given by

$$\left\| \tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x) \right\| = O\left(\frac{1}{\eta^{\alpha - 1/r}} \right),\ 1/r < \alpha < 1. \tag{6.2}$$

3. If $\ell = 1$ in case (1) then the estimation of error of the signals (functions) belonging to $Lip\{\zeta(t), r\}$ class by $(C,2)(E,\ell)$ means is given by

$$\left\| \tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x) \right\| = O\left\{ \eta^{1/r} \zeta\left(\frac{1}{\eta} \right) \right\}. \tag{6.3}$$

4. If $r \to \infty$ in case (2), then $Lip(\alpha,\ r)$ class reduces to the class $Lip\,\alpha$, then the estimation of error of the signals (functions) by $(C,2)(E,\ell)$ means is given by

$$\left\| \tilde{\tilde{\Omega}}_n^{(C,\,2)(E,\,\ell)} - \tilde{\tilde{\chi}}(x) \right\| = O\left(\frac{1}{\eta^{\alpha - 1/r}} \right),\ 0 < \alpha < 1. \tag{6.4}$$

7. Acknowledgment

On of the author Krishna Kumar is grateful to CSIR, New Delhi, India for providing financial assistance in the form of Junior Research Fellowship (JRF) Ref.No. 18/06/2017(i) EU-V, (CSIR-UGC NET-2017).

References

[1] G.H.Hardy, Divergent series, Oxford University press, 1949.

[2] H.K.Nigam and K.Sharma, Degree of approximation of a class of function by (C,1)(E,q) mean of Fourier series, IAENG International Journal of Applied mathematics, 41:2, IJAM-2011.

[3] J.K. Kushwaha, L. Rathour, L.N. Mishra and R. Vishwakrma, On the degree of approximation of conjugate functions using generalized Nörlund-Euler summability method, GANITA, Vol. 72(2), (2022), 01-09.

[4] B.E.Rhaodes, On the degree of approximation of functions belonging to Lipschitz class by Housdroff means of its Fourier series, Tamkang J.Math. 34(2003), no.3, 245-247.

[5] J.K.Kushwaha, Approximation of function by (C,2)(E,1) product summability method of Fourier series, Ratio Mathematica, vol.38, (2020), pp 341-348.

[6] E.C.Titchmarsh, The Theory of functions, Oxford University press, 1949.

[7] Lal and Singh, Degree of approximation of conjugate of functions belonging to Lip(α,p) class by (C,1)(E,1) means of conjugate series of Fourier series, Tamkang Journal of Mathematics, Vol.33, No.1(2002), 13-26.

[8] S.Lal, Approximation of functions belonging to the generalized Lipschitz class by $(C^1,.N_p)$ summability method of Fourier series, Applied Mathematics and Computation, 209(2), 346-350, (2009).

[9] B.N. Sahney and D.S. Goel, On the degree of approximation of functions belonging to Lipschitz class by Housdorff means of its Fourier series, Tamkang J.Math, 4(2003), 245-247.

[10] K. Qureshi, On the degree of approximation of a function belonging to Weighted class W(ξ(t),r), Indian J. pure Appl. Math. 13(1982), 898-903.

[11] L.N. Mishra, V.N. Mishra and V. Sonavane, Trigonometric approximation of functions belonging to Lipschitz class by matrix $(C^1,.N_p)$ operator of conjugate series of Fourier series, Advances in Difference Equations (2013) 2013:127.

[12] K. Qureshi, On the degree of approximation of function belonging to the class Lipα, Indian J. pure Appl. Math. 13 No.8(1982), 898-903.

[13] V.N. Mishra, K. Khatri and L.N. Mishra, Product $(N_p.C^1)$ summability of a sequence of Fourier coefficients, Mathematical Sciences – a Springer open Access Journal, (2012) 6:38.

[14] V.N.Mishra, K.Khatri and L.N.Mishra, Approximation of functions belonging to Lipschitz class by $(C^1,.N_p)$ summability Method of conjugate series, Mathematicki Vesnik, 66(2014), no. 2, 155-164.

[15] H.K.Nigam, A study on approximation of conjugate of functions belonging to Lipschitz class and generalized Lipschitz class by product means of conjugate series of Fourier series, Thai J. math., 10(2012), no.2, 275-287.

[16] K.Qureshi and H.K.Nema, A class of functions and their degree of approximation, Ganita, 41, No.1, (1990), 37-42.

[17] A.Zygmund, Trigonometric Series, 2nd edn., Vol.1, Cambridge University Press, Cambridge, (1959).

[18] Làszló Leindler, Trigonometric approximation of functions in L^p norm, J.Math. Anal Appl. 302(2005).

[19] J.K.Kushwaha and K. Kumar, On the approximation of function of conjugate of function belonging to the generalized Lipschitz class by Euler-matrix product summability method of conjugate series of Fourier series, Ratio Mathematica, Volume 42, (2022).

[20] Deepmala, Piscoran Laurian-Ioan, Approximation of signals (functions) belonging to certain Lipschitz classes by almost Riesz means of its Fourier series, Journal of Inequalities and Applications, 2016, 2016:163. DOI: 10.1186/s13660-016-1101-5

[21] L.N. Mishra, V.N. Mishra, K. Khatri, Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi (t)) (r \geq 1)-$ class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, Applied Mathematics and Computation, Vol. 237 (2014) 252-263.

[22] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, Trigonometric approximation of periodic Signals belonging to generalized weighted Lipschitz $W' (L_r, \xi(t)), (r \geq 1)-$ class by N\"{o}rlund-Euler $(N, p_n) (E, q)$ operator of conjugate series of its Fourier series, Journal of Classical Analysis, Volume 5, Number 2 (2014), 91-105. doi:10.7153/jca-05-08.

[23] Deepmala, L.N. Mishra, V.N. Mishra; Trigonometric Approximation of Signals (Functions) belonging to the $W (L_r, \xi(t)), (r \geq 1)-$ class by (E, q) (q > 0)-means of the conjugate series of its Fourier series, GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13, Global Journal of Mathematical Sciences, Vol 2. No. 2, pp. 61–69, (2014).

[24] L.N. Mishra, M. Raiz, L. Rathour, V.N. Mishra, Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces, Yugoslav Journal of Operations Research, Vol. 32, No. 3, (2022), 377-388. DOI: https://doi.org/10.2298/YJOR210915005M

[25] V.N. Mishra, K. Khatri, L.N. Mishra, Using Linear Operators to Approximate Signals of Lip (α, p), (p≥1)-Class, Filomat, 27:2 (2013), 353-363, DOI 10.2298/FIL1302353M,

[26] V.N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee 247 667, Uttarakhand, India.

[27] V.N. Mishra, L.N. Mishra, Trigonometric Approximation of Signals (Functions) in Lp− norm, International Journal of Contemporary Mathematical Sciences, Vol. 7, no. 19, 2012, pp. 909 – 918.