

# A Modified Constrained Optimal Replacement Model for Nonlinearly Deteriorating Systems under Limited Resources

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**Abstract:-** Optimal replacement of failing systems is a classical problem in asset management, operation research and reliability engineering. The assumption of classical replacement model such as linear deterioration and unconstrained decision contexts rarely hold in real industrial practice. A modified restricted optimal replacement model is proposed in this study to explicitly incorporate operational and financial restrictions into the replacement decision making. System performance deteriorates following a nonlinear dynamic process. Replacement actions are represented as binary cost variables for a continuous-time optimal control optimal control system. Convex operating and maintenance costs are introduced to penalize extreme deterioration. The existence and uniqueness of solutions, positive invariance, global asymptotic stability of the degradation dynamics, and boundedness of the cost functional are among the rigorous theoretical properties of the model that are established. Sensitivity analysis shows how the curve of maintenance cost, replacement cost, and degradation intensity affect the best time to replace. Numerical simulation validate the suggested framework's practical applicability. The model offers a tool for restricted replacement planning in industrial systems that is both realistically implementable and mathematically sound.

**Keywords:** Optimal replacement, Nonlinear deterioration, Resources constraint, Optimal Control, Stability analysis, Asset management.

## 1. Introduction

Industrial systems such as manufacturing machinery, power generation equipment and infrastructure assets inevitably degrade over time due to aging, wear and operational stress. In an effort to reduce infrastructure life cycle costs while maintaining acceptable levels of performance and reliability, the best time to replace systems must be determined. Traditional replacement models, developed using dynamic programming and renewal theory, typically presume unrestricted choice environments. Implicit in these models is the assumption of unrestricted downtime, unlimited finances, and sufficient staff. But in real industrial applications such assumptions are hardly met as choices on replacement are limited by budgetary constraints, operational downtime constraints and availability of human resources. Failure to take these limitations into account will lead to theoretically ideal but practically impossible replacement policies.

This paper creates a restricted optimal replacement paradigm that incorporates nonlinear deterioration dynamics with explicit resource limits, driven by this gap between theory and reality. The suggested formulation fills the gap between practical industrial decision-making and traditional replacement theory.

## 2. Literature Review

In 1976, Pierskalla and Voelker [12] gave a model which involves an optimal decision to procure, inspect, and repair and/or replace a unit subject to deterioration in service. An assumption that offers a technical convenience was introduced by Block and Borges (1985)[4]. It is unrealistic in many circumstances. It would be more realistic if the unit was fixed and returned to its pre-failure state. An outline of maintenance optimization model uses was provided by Dekker (1996)[6]. He examined how these models functioned in maintenance and talked about potential obstacles to their use. Cassady and Kutanoglu (2005)[5] conducted a comprehensive experimental investigation employing modest scheduling problems to examine the benefits of combining production scheduling with preventative maintenance planning. They contrasted the effectiveness of the integrated approach with the results of resolving the task scheduling and preventive maintenance planning issues separately. Cong Yang & Hao Zhang (2024)[18] studied the optimal maintenance of a machine whose state is unobservable and deteriorates according to a discrete-time finite-state Markov process and developed a policy iteration algorithm to find the optimal solution exactly in finite time. Nobuyuki Tamura (2023)[14] considered a system whose deterioration follows a discrete-time and discrete-state Markov chain with an absorbing state. And proposed a POMDP model and theoretically investigate the structure of an optimal maintenance policy minimizing a total expected discounted cost for an unbounded horizon. Then two Oke, (2024)[11] developed optimal maintenance policies for industrial machines to ensure uninterrupted production. Analytical and intelligent replacement models were constructed using real-time data from a commercial photocopier operator's expenditure logbook between 2012 and 2023. Zhang (2024)[19] proposed the environmental protection gas switchgear data processing technology based on data enhancement technology. In his work he proposed the environmental protection gas switchgear life prediction technology based on approximate dynamic planning, and verifies the algorithm through case analysis, which proves the effectiveness and accuracy of the proposed method. Geurtsen M. et al.(2025)[7] shown that the proposed DRL approach consistently outperforms traditional dispatching rules and practitioner-designed heuristics. This study is the first to apply DRL to maintenance decision-making in serial production systems that explicitly model product quality degradation, machine production states and buffer interactions addressing a key gap in current literature. In the domains of operations research, reliability engineering, and industrial management, the issue of the best way to replace and maintain failing systems has garnered a lot of interest. Numerous mathematical models have been created throughout the years to identify the best practices for maintaining or replacing systems that deteriorate over time. The assumptions made by these models about resource limitations, uncertainty, degradation behavior, and decision-making frameworks vary. Following these foundational contributions, researchers began to incorporate more realistic assumptions into replacement and maintenance models. Block et al. (1985)[4] introduced age-dependent minimal repair models, which account for the changing failure behavior of systems over time. Nakagawa (1988)[14] further extended the theory by developing preventive maintenance and replacement policies, including age replacement and block replacement strategies. These models improved practical applicability but still relied on simplified deterioration structures and did not explicitly consider constraints such as budget or resource limitations. In the 1990s and early 2000s, significant system states and observations. Alaswad and Xiang (2023)[1] reviewed condition-based maintenance optimization models, emphasizing stochastic deterioration processes and real-time monitoring. Similarly, Yang and Zhang (2024)[18] studied optimal control of Markov deterioration processes, integrating stochastic dynamics with control-based decision-making. The emergence of artificial intelligence and machine learning has further expanded the scope of replacement and maintenance modeling. Oke et al. (2024)[11] proposed intelligent preventive replacement policies for non-repairable systems, incorporating data-driven techniques for improved decision-making. Geurtsen et al. (2025)[7] applied deep reinforcement learning to maintenance optimization, enabling adaptive and automated policy generation. These approaches offer significant advantages in handling complex and uncertain environments but often lack analytical transparency and require large datasets for implementation. Other recent studies have explored simulation-based and hybrid approaches. Monte Carlo simulation models (2025) are widely used to evaluate maintenance strategies under uncertainty, while multi-component system models such as Barde et al. (2025)[3] address the complexity of interconnected systems. Additionally, reliability-based models published in recent journals (2023–2024) incorporate nonlinear deterioration and probabilistic failure behavior, further improving

the realism of replacement decisions. Despite these advancements, several gaps remain in the literature. Many classical models assume linear deterioration and ignore practical constraints, limiting their applicability. Modern stochastic and AI-based approaches, while powerful, often involve high computational complexity and lack closed-form analytical solutions. Furthermore, relatively few studies explicitly integrate nonlinear deterioration, resource constraints, and optimal control within a single unified framework for replacement decision-making. Motivated by these limitations, the present study proposes a modified constrained optimal replacement model for systems undergoing nonlinear deterioration under limited resources. Unlike existing approaches, the proposed model combines nonlinear deterioration dynamics with budgetary and operational constraints, using an optimal control framework to determine replacement decisions. This integration provides a balance between theoretical rigor and practical applicability, offering a more comprehensive solution to real-world replacement problems. In summary, the literature on replacement and maintenance modeling has evolved from simple deterministic models to complex stochastic and data-driven approaches. However, the need for analytically tractable models that incorporate realistic constraints and nonlinear system behavior remains significant. The proposed study contributes to this ongoing research by addressing these gaps and advancing the state of the art in replacement modelin advancements were made in maintenance optimization. Dekker (1996)[6] reviewed various maintenance optimization models and emphasized the importance of cost-based decision-making. Wang (2002)[15] provided a comprehensive survey of maintenance policies for deteriorating systems, including both deterministic and stochastic approaches. These studies introduced more sophisticated modeling techniques, such as probabilistic failure modeling and condition-based maintenance, thereby enhancing the realism of replacement decisions. However, while some of these models considered uncertainty and partial constraints, they generally did not integrate all aspects of real-world decision-making into a unified framework. Further developments in the field focused on integrating maintenance planning with operational decision-making. Cassady and Kutanoglu (2005)[5] studied the interaction between preventive maintenance and production scheduling, demonstrating the importance of coordinated decision-making in industrial systems. Jardine and Tsang (2006)[8] presented a comprehensive treatment of maintenance, replacement, and reliability theory, incorporating both theoretical and practical aspects. Zhou et al. (2007)[17] introduced predictive maintenance models based on reliability-centered approaches, which utilize system condition and failure prediction to optimize maintenance schedules. While these studies significantly advanced the field, they primarily focused on maintenance strategies rather than explicit replacement decisions under constraints. The introduction of optimal control theory marked a major shift in the modeling of dynamic systems. Sethi and Thompson (2000)[13] demonstrated how optimal control techniques can be applied to management science and engineering problems, including maintenance and replacement. These methods allow for continuous-time decision-making and provide a powerful framework for handling dynamic systems. However, control-theoretic approaches often focus on general optimization problems and may not explicitly address replacement policies in constrained environments. In recent years, research has increasingly focused on stochastic modeling and data-driven approaches. Tamura (2023)[14] analyzed maintenance problems using partially observable Markov decision processes (POMDPs), capturing uncertainty in system states and observations. Alaswad and Xiang (2023)[1] reviewed condition-based maintenance optimization models, emphasizing stochastic deterioration processes and real-time monitoring. Similarly, Yang and Zhang (2024)[18] studied optimal control of Markov deterioration processes, integrating stochastic dynamics with control-based decision-making. The emergence of artificial intelligence and machine learning has further expanded the scope of replacement and maintenance modeling. Oke et al. (2024)[11] proposed intelligent preventive replacement policies for non-repairable systems, incorporating data-driven techniques for improved decision-making. Geurtsen et al. 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behavior, further improving the realism of replacement decisions. Despite these advancements, several gaps remain in the literature. Many classical models assume linear deterioration and ignore practical constraints, limiting their applicability. Modern stochastic and AI-based approaches, while powerful, often involve high computational complexity and lack closed-form analytical solutions. Furthermore, relatively few studies explicitly integrate nonlinear deterioration, resource constraints, and optimal control within a single unified framework for replacement decision-making. Motivated by these limitations, the present study proposes a modified constrained optimal replacement model for systems undergoing nonlinear deterioration under limited resources. Unlike existing approaches, the proposed model combines nonlinear deterioration dynamics with budgetary and operational constraints, using an optimal control framework to determine replacement decisions. This integration provides a balance between theoretical rigor and practical applicability, offering a more comprehensive solution to real-world replacement problems. In summary, the literature on replacement and maintenance modeling has evolved from simple deterministic models to complex stochastic and data-driven approaches. However, the need for analytically tractable models that incorporate realistic constraints and nonlinear system behavior remains significant. The proposed study contributes to this ongoing research by addressing these gaps and advancing the state of the art in replacement modelling.

### 3. Research Gap

The existing literature reveals the following gaps:

- Absence of unified nonlinear frameworks integrating continuous deterioration discrete replacement actions, and multiple resource constraints.
- Limited theoretical analysis of stability and feasibility in constrained replacement systems.
- Insufficient sensitivity analysis of optimal replacement timing with respect to key parameters.

The current study fills in these gaps with a formulation that is both realistically oriented and mathematically precise.

### 4. Motivation and Background

Decisions for industrial replacement are rarely unrestricted. Power plants are subject to harsh fines for excessive downtime, manufacturing facilities are subject to set annual maintenance budgets, and transportation infrastructure relies on a shortage of specialized labor for replacement operations. A replacement model that is both operationally feasible and analytically tractable is required in light of these circumstances.

The proposed model is designed to:

- i. Capture nonlinear deterioration behavior.
- ii. Reflect escalating maintenance costs through convex cost structures,
- iii. Explicitly incorporate budget and operational constraints.
- iv. Remain amenable to stability, sensitivity, and optimal control analysis.

### 5. Methodology

#### 5.1 Optimal Control Problem

The basic of optimal control problem is a control system. It is describe by ordinary differential equation of the form

$$\frac{dx}{dt} = f(t, x, u) \quad \text{with} \quad x(t_0) = x_0$$

where  $x$  is some value in  $R^n$ .  $u$  is control input and  $u \in U \subseteq R^m$  where  $U$  is control set.  $t$  is time and  $t_0$  is initial value  $x = x(t)$  &  $u = u(t)$ . And the cost functional is defined by

$$J(u) = \int_{t_0}^{t_f} L(t, x, u) dt + K(t_f, x_f)$$

initially at  $x(t_0) = x_0$  where  $L: R \times R_n \times U \rightarrow R$  is running cost and  $K: R \times R_n \rightarrow R$  is terminal cost. The final time is denoted by  $t_f$  which is either free or fixed and  $x_f = x(t_f)$  is known as state which is also either fixed or free or some given target set and  $u$  is a function of time.

If the cost is given then such optimal control problem are known as problem in the Bolza form or Bolza problem.

There are two special cases:

- i. If there is no terminal cost i.e.  $K = 0$  then such problem is known as Lagrange problem.
- ii. And if the running cost is zero then such type of problem is known as Mayer problem. For the Bolza problem this gives

$$J(u) = \int_{t_0}^{t_f} L(t, x, u) dt + K(t_f, x_f) \quad \text{with } x \in R^{n+1}$$

Now for concreteness, we suppose problem with fixed time and free end point i.e. the target set is  $S = \{t_1 \times R^n\}$ . Then the cost functional defined as

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt + K(x(t_1))$$

Here  $K(x(t_1))$  is taken as zero i.e. we consider it as Lagrange problem. Then above equation written as

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt$$

Then the minimization problem as

$$J(u) = \int_{t_0}^{t_1} L(s, x(s), u(s)) ds \quad (1)$$

Where  $t \in [t_0, t_1]$  and  $x \in R^n$ , here  $x$  stands for the state trajectory corresponding to the control  $u$  and satisfy  $\dot{x}(t) = f(t, x, u)$ .

Introducing the optimal function now

$$O(t, x) = \inf_{u[t, t_1]} J(t, x, u) \quad (2)$$

Where  $u[t, t_1]$  tells that  $u$  is restricted to the interval  $[t, t_1]$ . The optimal function must satisfy the boundary condition

$$V(t, x) = 0 \quad (3)$$

Thus for every  $(t, x) \in [t_0, t_1] \times R^n$  and every  $\Delta t \in [0, t_1 - t]$ . The optimal function  $O$  satisfies the relation

$$O(t, x) = \inf_{u[t, t_1]} \int_t^{t+\Delta t} L(s, x(s), u(s)) ds + O(t + \Delta t, x(t + \Delta t)) \quad (4)$$

Where  $x(t + \Delta t)$  is the state trajectory corresponding to the control  $u[t, t + \Delta t]$  and satisfy  $\dot{x}(t) = f(t, x, u)$ .

Now we search small time interval for a control that minimizes the cost. Thus the minimization problem on the interval  $[t, t_1]$  is divided into two, first one on  $[t, t + \Delta t]$  and the other on  $[t + \Delta t, t_1]$ .

The optimality principle mentioned above might seem apparent. However, since we are utilizing an infimum and do not assume the existence of optimal controls, it is crucial to thoroughly defend it. We provide “one half” of the evidence by confirming that

$$O(t, x) \geq \bar{O}(t, x) \quad (5)$$

Where

$$\bar{O}(t, x) = \inf_{u[t, t_1]} \int_t^{t+\Delta t} L(s, x(s), u(s)) ds + O(t + \Delta t, x(t + \Delta t))$$

By equation (2) and definition of infimum, there exist a control  $u_\epsilon$  on  $[t + t_1]$  for every  $\epsilon > 0$  such that

$$O(t, x) + \epsilon = J(t, x, u_\epsilon) \quad (6)$$

$x_\epsilon$  for the corresponding state trajectory

$$\begin{aligned} J(t, x, u_\epsilon) &= \int_t^{t+\Delta t} L(s, x_\epsilon, u_\epsilon) ds + J(t + \Delta t, x_\epsilon(t + \Delta t), u_\epsilon) \\ &\geq \int_t^{t+\Delta t} L(s, x_\epsilon, u_\epsilon) ds + O(t + \Delta t, x_\epsilon(t + \Delta t)) \geq \bar{O}(t, x) \end{aligned}$$

Where the inequalities follow directly from the definition of  $O$  and  $\bar{O}$  respectively. Since equation (6) holds with an arbitrary  $\epsilon > 0$ , then the desired inequalities (5) is established.

## 6. Assumption and Notations of the Proposed Model

### Assumptions

#### Continuous Deterioration:

Continuous deterioration refers to a progressive and uninterrupted degradation process in which the performance, efficiency, or reliability of a system, component, or inventory item declines smoothly over time. This degradation is typically modeled as a continuous function of time, implying the absence of abrupt changes or discrete jumps in the deterioration path.

Let  $x(t)$  denote the state (e.g., performance level, strength, or quality) of a system at time  $t \geq 0$ . The system is said to undergo continuous deterioration if:

$$\frac{dx(t)}{dt} > 0, \quad \forall t > 0$$

and  $x(t)$  is a continuous (often differentiable) function of time

This implies that the system's state decreases monotonically and smoothly as time progresses.

#### Nonlinear Deterioration Rate:

A degradation process where the rate of deterioration of a system, component, or inventory item is a nonlinear function of time, system state, or both is referred to as a nonlinear deterioration rate. Nonlinear degradation captures more intricate and realistic dynamics like acceleration, deceleration, or state-dependent wear, as contrast to linear deterioration, where the rate is constant or proportionate in a straightforward manner.

Let  $x(t)$  denote the state of the system at time  $t \geq 0$ . The system is said to follow a nonlinear deterioration rate if its evolution satisfies:

$$\frac{dx(t)}{dt} = -g(t, x(t)), \quad \forall t > 0$$

where  $g(t, x(t))$  is a nonlinear function in  $t$ ,  $x(t)$  or both, and  $g(t, x(t)) > 0$ .

### Irreversibility without Replacement:

When a system's deterioration or degradation process is strictly one-way and cannot be stopped, fixed, or restored, and when no replacement activity is made to renew the system, it is said to be irreversible without replacement. Consequently, the system keeps deteriorating until it hits an undesirable performance barrier or a breakdown state.

Let  $x(t)$  denote the state of the system at time  $t \geq 0$ . The system exhibits irreversibility without replacement if:

The state function is monotonically decreasing:

$$x(t_2) \leq x(t_1) \quad \forall t_2 > t_1$$

Once degraded, the system cannot return to any previous higher state:

$$x(t + \Delta t) \leq x(t)$$

### Perfect Replacement:

A maintenance or replacement policy known as "perfect replacement" ensures that a deteriorating system is instantly restored to "as-good-as-new" state upon replacement. Following replacement, all previous degradation effects are totally removed, and the system's state, performance, and failure characteristics are exactly the same as those of a brand-new unit.

Let  $x(t)$  denote the state of the system at time  $t$  and  $T$  let be the replacement time. Under perfect replacement, the system satisfies:

$$x(T) \leq x_0$$

where:

$x(T)$ : state immediately after replacement

$x_0$ : initial state of the system

### Convex Maintenance Cost:

The term "convex maintenance cost" describes a cost structure where a system's maintenance or repair costs are a convex function of time, system age, or degree of deterioration. This suggests that as the system deteriorates the marginal cost of maintenance rises, reflecting the increasing complexity and cost of maintaining an old or degraded system.

Let  $C_m(x)$  be the maintenance cost as a function of the system state  $x$  (e.g., deterioration level or age). The cost function is said to be convex if:

$$\frac{d^2 C_m(x)}{dx^2} \geq 0 \quad \forall x$$

### Resource Constraints:

The term "resource constraints" describes restrictions on the availability and distribution of necessary resources, such as funds, personnel, time, materials, or capacity, that limit the range of choices that can be made in a system, optimization problem, or operational procedure. When choosing the best maintenance, replacement, or control strategies, these limitations are crucial.

## Notations

| Symbol    | Description                           |
|-----------|---------------------------------------|
| $t$       | Continuous time variable              |
| $x(t)$    | System performance/state at time $t$  |
| $x_0$     | Initial (new) system performance      |
| $f(x)$    | Nonlinear deterioration function      |
| $u(t)$    | Replacement control variable (0 or 1) |
| $C_m$     | Operating and maintenance cost        |
| $C_r$     | Fixed replacement cost                |
| $B$       | Available maintenance cost            |
| $T$       | Replacement horizon                   |
| $T_{max}$ | Maximum allowable Time                |
| $J(u)$    | Total Cost functional                 |

## 7. Proposed Model

Richard Bellman (1957) introduced the principle of optimality, formulating replacement problems through dynamic programming

$$\frac{d^2 C_m(x)}{dx^2} \geq 0 \quad \forall x$$

This discrete-time framework does not capture continuous system deterioration. The introduction of the Pontryagin Maximum Principle by Lev Pontryagin (1962) enabled the formulation of replacement problems as continuous time optimal control system

$$\dot{x}(t) = f(x(t), u(t))$$

$$J(u) = \int_0^T L(x(t), u(t)) dt$$

This approach allows dynamic interaction between state and decisions variables but usually assumes simplified system behaviour.

Toshio Nakagawa (2005) Capture realistic degradation & proposed a nonlinear deterioration models

$$\dot{x} = -\alpha x^\alpha \quad \text{where } x = x(t)$$

This models successfully represent accelerated degradation.. In the view of above research gap, this study develops a continued nonlinear optimal control model that integrates

- i.nonlinear deterioration dynamics
- ii.binary replacement decisions
- iii.convex maintenance cost structure
- iv.Explicit resource constraints.

Then the system is modeled using a continuous-time dynamic- framework where the state variable represents

system performance & control variable represents replacement actions.

### 7.1 State Dynamics

The system evolves according to the controlled nonlinear differential equation

$$\dot{x}(t) = f(x(t) - u(t)x(t), \quad x(0) = x_0 > 0$$

Where

$$u(t) \in [0,1], \quad t \in [0, T]$$

A commonly adopted form of deterioration is

$$f(x) = -\alpha x^\gamma, \quad \alpha > 0, \gamma \geq 1$$

Which capture accelerated degradation at lower performance levels.

### 7.2 Cost Functional

The total cost over the planning horizon  $[0, T]$  is

$$\text{Min } J(u) = \int_0^T [C_m(x(t)) + C_r u(t)] dt$$

Where  $C_m(x) = c_0 + c_1(1 - x)^\beta$ ,  $c_0, c_1 > 0, \beta \geq 1$

And subject to constraint

$$\int_0^T C_m(x(t)) dt + C_r \int_0^T u(t) dt \leq B$$

And

$$T \leq T_{\max}$$

### Optimal Control Policy

$$\text{Min } J(u) = \int_0^T [C_m(x(t)) + C_r u(t)] dt$$

And subject to constraint

$$\dot{x}(t) = -\alpha x^\gamma - u(t)x(t), \quad x(0) = x_0 > 0$$

$$\int_0^T C_m(x(t)) dt + C_r \int_0^T u(t) dt \leq B$$

Let  $\lambda$  is Lagrange multiplier

$$\begin{aligned} J^*(u) &= \int_0^T [C_m(x) + C_r u + \lambda(C_m(x) + C_r u)] dt \\ &= \int_0^T (1 + \lambda)[C_m(x) + C_r u] dt \end{aligned}$$

Now we introduce Hamiltonian Function

$$H = (1 + \lambda)(C_m(x) + C_r u) + p(t)(-\alpha x^\gamma - ux)$$

Necessary Condition

1. State Equation

$$\dot{x} = -\alpha x^\gamma - ux$$

2. Costate Equation

$$\begin{aligned}\dot{p} &= -\frac{\partial H}{\partial x} \\ &= -(1 + \lambda)C'_m(x) + p(\alpha\gamma x^{\gamma-1} + u)\end{aligned}$$

### 3. Optimal Control Condition

$$\begin{aligned}\frac{\partial H}{\partial u} &= 0 \\ &= (1 + \lambda)C_r - px = 0 \\ p &= \frac{(1 + \lambda)}{x}C_r\end{aligned}$$

Since  $u(t) \in 0, 1$  Then Hamiltonian is linear in terms of  $u$ . This implies (Bang-Bang Control) In this control (Decision Rule)

If

$$(1 + \lambda)C_r - px > 0 \Rightarrow u = 0$$

If

$$(1 + \lambda)C_r - px < 0 \Rightarrow u = 1$$

From here we can interpolated following result

- If system deterioration is high then replacement is optimal
- If cost is higher there is delay replacement then we get the solution

$$u(t) = \begin{cases} 0, & t < t^* \\ 1, & t \geq t^* \end{cases}$$

where  $t^*$  is switching time.

### The role of budget constraint:

- If the budget is tight then there is delay replacement
- If we have enough budget then replacement is fast.  $\lambda$  always control it.

Here  $p(t)$  is called cost variable basically  $p(t)$  is marginal value (shadow price) of  $x(t)$ . It mean if we change  $x(t)$  slightly then it represent how much total cost  $J$  change. i.e.

$$p(t) = \frac{\partial J}{\partial x(t)}$$

i.e.  $x(t)$  is small then effect on total cost is  $p(t)$ . If  $p(t)$  is large, system is valuable & in this case replacement is not necessary i.e.  $u = 0$ .

If  $p(t)$  is small then system is not useful & the cost is high in future & then replace it i.e.  $u = 1$

Since our switching condition was

$$(1 + \lambda)C_r = p(t)x(t)$$

where  $C_r$  is replacement cost.

$p(t)x(t)$  is future value of system maintenance.

### Economic Interpretation:

If

$$p(t)x(t) > (1 + \lambda)Cr$$

In this case the system is valuable this implies there is no replacement.

If

$$p(t)x(t) < (1 + \lambda)Cr$$

In this case the system is useless this implies there is replacement.

### Real-life interpretation

If  $x(t)$  is machine performance &  $p(t)$  is future earning capacity of machine. Then  $p(t)x(t)$  future benefit from machine.  $C_r$  is cost of new machine.  $p(t)$  indicates whether it is more beneficial to save the system or to replace it.

### 8. Stability Analysis

The stability analysis of the proposed nonlinear replacement model is carried out to investigate the long-term behavior of the deteriorating system and to verify whether the proposed control policy guarantees bounded and stable system performance. The analysis consist of equilibrium analysis, local stability, global asymptotic stability, exponential stability and stability under optimal replacement control.

The proposed state equation is

$$\dot{x} = -\alpha x^\gamma - ux$$

Equilibrium Points

For equilibrium state

$$\dot{x} = 0$$

Hence

$$-\alpha x^\gamma - ux = 0$$

$$-x(\alpha x^{\gamma-1} + u) = 0$$

$$x^* = 0, \left(-\frac{u}{x}\right)^{\frac{1}{\gamma-1}}$$

Since  $\alpha > 0, u \geq 0, \gamma \geq 1$  no positive equilibrium exist.

$\Rightarrow x^* = 0$  is the unique equilibrium point.

Thus the system always converges toward the deterioration state unless replacement restores the systems.

### Local Stability Analysis

Let

$$f(x) = -\alpha x^\gamma - ux$$

The Jacobian matrix is

$$J(x) = \frac{df}{dx} = -\alpha \gamma x^{\gamma-1} - u$$

$$J(x^*) = -\alpha \gamma (x^*)^{\gamma-1} - u$$

**Case I: Linear deterioration ( $\gamma = 1$ )**

$$J(x^*) = -\alpha - u$$

$$J(0) = -(\alpha + u)$$

Eigenvalue,

$$\lambda = -(\alpha + u)$$

$$\Rightarrow \lambda < 0$$

Therefore,  $x^* = 0$  is locally asymptotically stable.

**Case II: Nonlinear deterioration ( $\gamma > 1$ )**

$$J(x^*) = -\alpha\gamma(x^*)^{\gamma-1} - u$$

$$J(0) = -u$$

Which implies asymptotic stability.

$$u = 0,$$

$$J(0) = 0$$

$$\lambda = 0$$

Linearization becomes inconclusive because

$$J(0) = 0$$

Therefore, nonlinear stability methods are required.

**Lyapunov Stability Analysis**

To establish global stability, consider the Lyapunov function

$$V(x) = \frac{1}{2}x^2$$

Clearly,

$$V(x) > 0, \quad \forall x_0 \neq 0,$$

And

$$V(0) = 0.$$

The derivative along system trajectories is  $\dot{V} = x\dot{x}$  substituting the system equation,

$$\dot{V} = x(-\alpha x^\gamma - ux)$$

Hence,

$$\dot{V} = -\alpha x^{\gamma+1} - ux^2$$

$$\dot{V} \leq 0$$

Moreover,  $\dot{V} = 0$  only when  $x = 0$ .

Hence  $V(x)$  is positive definite while  $\dot{V}$  is negative definite.

According to Lyapunov's Direct Method.

$x = 0$  is globally asymptotically stable.

Because  $\dot{V} \leq 0$  every trajectory remains inside the compact level set.

$$\Omega = \{x: V(x) \leq c\}.$$

The invariant set satisfying  $\dot{V} = 0$  is  $\{0\}$ .

Applying LaSalle's Invariance Principle, every trajectory satisfies  $t \rightarrow \infty \quad \lim x(t) = 0$ .

Therefore,  $x(t) \rightarrow 0$  globally.

### Exponential Stability

For  $\gamma = 1$ , the state equation becomes

$$\dot{x} = -(\alpha + u)x$$

It's solution is

$$x(t) = x_0 e^{-(\alpha + u)t}$$

Hence

Therefore, the equilibrium is globally exponentially stable.

The exponential convergence rate equals.

### Stability under Optimal Replacement Control:

The optimal control policy is

$$u(t) = \begin{cases} 0, & t < t^* \\ 1, & t \geq t^* \end{cases}$$

Before replacement

$$\dot{x} = -\alpha x^\gamma$$

After replacement

$$\dot{x} = -\alpha x^\gamma - x$$

The Jacobian after replacement becomes

$$\dot{x}(t) = -\alpha \gamma x^{\gamma-1} - 1$$

Since  $J > 0 \quad \forall x > 0$ , replacement increases the magnitude of the negative eigenvalue.

Therefore, the closed loop system converges faster after replacement.

Hence, optimal replacement improve system stability.

### Boundedness of the State Variable

Since

$$\dot{x} = -\alpha x^\gamma - ux \leq 0$$

The state variable is monotonically decreasing.

Therefore  $0 \leq x(t) \leq x_0$ .

Therefore  $x(t)$  remains bounded for all  $t \geq 0$ .

**Theorem 8.1 (Existence and Uniqueness)**

For any admissible control  $u(t)$ , the state equation  $\dot{x}(t)$  admits a unique positive solution on  $[0, T]$ .

**Proof:** Assume that the state equation is:

$$\dot{x}(t) = f(x(t), u(t)), x(0) = x_0 > 0$$

Where  $f$  is locally Lipschitz in  $x$  and continuous.  $u(t)$  is measurable and bounded since it is admissible. As a result, the function  $f(t; x) = f(x, u(t))$  is Lipschitz continuous locally in  $x$ .

According to the Picard-Lindelof theorem, the Initial Value Problem admits a unique solution on some interval for any initial condition  $x_0$ . The solution covers all of  $[0, T]$  since the right hand side of above equation is bounded on finite intervals.

Thus a unique solution exist on  $[0, T]$ .

**Theorem 8.2 (Positively Invariance)**

If  $x(0) > 0$  then  $x(t) > 0$  for all  $t \in [0, T]$

**Proof:** Assume the vector field satisfies

$$f(0, u(t)) \geq 0$$

(i.e. the points inward at the boundary  $x = 0$ )

Support for contradiction that the solution reaches zero. Then there exists a first time.

$$t^* = \inf\{t > 0; x(t) = 0\}$$

By continuity,  $x(t^*) = 0$  and  $x(t) > 0$  for all  $t < t^*$

At  $t = t^*$ ,

$$\dot{x}(t^*) = f(0, u(t^*)) \geq 0$$

Hence the trajectory cannot cross into the negative region. This contradicts the assumption that  $x(t)$  becomes negative after  $t^*$ . Therefore  $x(t) > 0 \quad \forall t$ . Since the dynamics are smooth, strict positivity is preserved.

$$x(t) > 0 \quad \forall t \in [0, T]$$

**Theorem 8.3 (Global Asymptotic Stability without Replacement)**

In the absence of replacement ( $u(t) = 0$ ), the equilibrium  $x = 0$  is globally asymptotically stable.

**Proof:** Assume  $\dot{x} = -f(x)$ ,  $f(x) > 0$  for  $x > 0$  Choose the Lyapunov function

$$V(x) = x, \quad \forall x > 0, V(0) = 0$$

Its derivative along trajectories is

$$\dot{V}(x) = \frac{d}{dt}x = -f(x)$$

Since  $f(x) > 0, \forall x > 0$

$$\dot{V}(x) < 0, \forall x > 0$$

Therefore  $V$  is positive definite and strictly decreasing.

By Lyapunov direct method, the equilibrium  $x = 0$  is globally asymptotically stable.

#### Theorem 8.4 (Global Stability)

Let then the equilibrium point  $x^* = 0$  of the nonlinear combined replacement model  $\dot{x} = -\alpha x^\gamma - u(t)x$  is globally asymptotically stable.

**Proof:**

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V}(x) = x\dot{x}$$

$$\dot{V}(x) = x[-\alpha x^\gamma - ux]$$

$$\dot{V}(x) = -\alpha x^{\gamma+1} - ux^2$$

$$\alpha > 0, u \geq 0$$

Therefore,  $\dot{V} < 0$  for every value of  $x$

#### 9. Sensitivity Analysis

Sensitivity analysis is an integral part of optimal replacement modeling as it studies the impact of uncertainties of model parameters on the optimal replacement policy and the total maintenance cost. In industrial systems, parameters like deterioration rate, maintenance cost, replacement cost, and the available budget are generally estimated from historical observations and contain unavoidable uncertainty. Hence, it is important to study the robustness of the proposed model to variations in the parameters.

The proposed constrained optimal replacement model is

$$\dot{x}(t) = -\alpha x^\gamma - ux$$

$$J(u) = \int_0^T [C_m(x(t)) + C_r u(t)] dt$$

Where  $C_m(x) = c_0 + c_1(1-x)^\beta$

The principal parameters governing the model are

$$\theta = \{\alpha, \gamma, c_0, c_1, \beta, C_r, B\}$$

$$J(u) = \int_0^T [C_m(x(t)) + C_r u(t)] dt \text{ Where } C_m(x) = c_0 + c_1(1-x)^\beta.$$

The principal parameters governing the model are

$$\theta = \{\alpha, \gamma, c_0, c_1, \beta, C_r, B\}$$

The influence of each parameter on the optimal replacement policy is examined individually while keeping the remaining parameter fixed.

#### Local Sensitivity Analysis

Suppose  $P = (P_1, P_2 \dots P_n)$  denotes the parameter vector.

The sensitivity of the objective function with respect to parameter  $P_i$  is

$$S_i = \frac{\partial J}{\partial P_i}$$

Since different parameters possess different physical units, normalized sensitivity indices are preferable.

The normalized sensitivity coefficient is

$$SI_i = \frac{\partial J}{\partial P_i} \frac{P_i}{J}$$

#### Interpretation:

- $SI_j > 0$ , Increases in parameter increases total cost.
- $SI_j < 0$ , Increases in parameter decreases total cost.
- Large absolute value, Parameter has strong influence.
- Small absolute value, Parameter has weak influence.

#### Sensitivity with respect to deterioration coefficient $\alpha$

The deterioration dynamics are

$$\dot{x} = -\alpha x^\gamma - u(t)x$$

The analytical solution without replacement is

$$x(t) = [x_0^{1-\gamma} + \alpha(\gamma-1)t]^{-\frac{1}{\gamma-1}}, \quad \gamma > 1$$

Differentiating with respect to  $\alpha$ .

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= -\frac{t}{[x_0^{1-\gamma} + \alpha(\gamma-1)t]^{-\frac{1}{\gamma-1}}} \\ &\Rightarrow \frac{\partial x}{\partial \alpha} < 0 \end{aligned}$$

An increase in deterioration intensity causes the system condition to decline faster.

Consequently,

- Maintenance cost increases
- Replacement occurs earlier
- Total life-cycle cost increases

Hence,

$$\frac{\partial J}{\partial \alpha} > 0$$

#### Sensitivity with respect to nonlinear exponent $\gamma$

The nonlinear exponent controls the curvature of deterioration.

Differentiating,

$$\frac{\partial \dot{x}}{\partial \gamma} = -\alpha x^\gamma \ln x$$

Therefore, when  $0 < x < 1, \ln x < 0$

Which accelerates deterioration during later stages.

A larger value of  $\gamma$ .

- Increases deterioration acceleration
- Shortens equipment lifetime,
- Advances replacement time,
- Increases maintenance expenditure.

#### Sensitivity with respect to maintenance cost parameter $c_1$

The maintenance cost function is

$$C_m(x) = c_0 + c_1(1-x)^\beta$$

$$\frac{\partial C_m}{\partial c_1} = (1-x)^\beta > 0$$

$$\frac{\partial J}{\partial c_1} = \int_0^T (1-x)^\beta > 0$$

#### Interpretation:

Higher maintenance cost immediately increases the objective function.

Consequently, the optimal controller prefers earlier replacement.

#### Sensitivity with respect to replacement cost parameter $C_r$

$$\frac{\partial J}{\partial C_r} = \int_0^T u(t) dt$$

$$\frac{\partial J}{\partial C_r} > 0$$

Engineering implication:

If replacement becomes expensive ,

the controller postpones replacement,

allowing additional maintenance

Therefore,

- Replacement frequency decreases ,
- Maintenance duration increases,
- Equipment operates longer before replacement.

#### Sensitivity with respect to budget $B$

The budget constraint is

$$\int_0^T C_m(x) dt + C_r \int_0^T u(t) dt \leq B$$

The associated Lagrange multiplier is  $\lambda$

Complementary slackness gives

$$\lambda(B - Cost) = 0.$$

Differentiating

$$\frac{\partial J}{\partial B} = -\lambda$$

$$\frac{\partial J}{\partial B} < 0, \quad \lambda \geq 0$$

Therefore, larger budget decreases the optimized cost.

#### Sensitivity with respect to convexity parameter $\beta$

Maintenance cost is

$$C_m(x) = c_0 + c_1(1-x)^\beta$$

Differentiation yields

$$\frac{\partial C_m}{\partial \beta} = c_1(1-x)^\beta \ln(1-x)$$

Thus the effect of  $\beta$  depends upon deterioration level.

Near failure,

Maintenance cost increases rapidly

Which shift the optimal replacement time towards earlier periods.

#### Sensitivity of switching Time

The optimal policy is

$$u(t) = \begin{cases} 0, & t < t^* \\ 1, & t \geq t^* \end{cases}$$

The switching time satisfies

$$(1 + \lambda)C_r = p(t^*)x(t^*)$$

Differentiating,

$$\frac{\partial t^*}{\partial \alpha} < 0, \frac{\partial t^*}{\partial c_1} < 0, \frac{\partial t^*}{\partial C_r} > 0, \frac{\partial t^*}{\partial B} < 0$$

Thus

- Increasing deterioration intensity advances replacement
- Increasing maintenance cost advances replacement
- Increasing replacement cost delays replacement
- Increasing budget advances replacement

### Normalized Sensitivity Index

For parameter  $P_i$

$$NSI_i = \frac{P_i}{J} \frac{\partial J}{\partial P_i}$$

Classification:

| NSI     | Interpretation       |
|---------|----------------------|
| >1      | Highly sensitive     |
| 0.5-1   | Moderately sensitive |
| 0.1-0.5 | Low sensitive        |
| <1      | Negligible           |

### Summary of Parameter Sensitivity:

| Parameter                     | Effect on Total cost | Effect on Replacement Time |
|-------------------------------|----------------------|----------------------------|
| Deterioration rate $\alpha$   | Increases            | Earlier                    |
| Nonlinear exponent $\gamma$   | Increases            | Earlier                    |
| Maintenance coefficient $c_1$ | Increases            | Earlier                    |
| Convexity parameter $\beta$   | Moderately increases | Earlier                    |
| Replacement cost $C_r$        | Increases            | Delayed                    |
| Available budget $B$          | Decreases            | Earlier                    |
| Planning horizon $T$          | Increases            | Depends on policy          |

The sensitivity study demonstrates that the deterioration coefficient  $\alpha$  and maintenance cost coefficient  $c_1$  are the dominant parameters influencing both the optimal replacement time and the total life-cycle cost. An increase in either parameter accelerates system degradation and raises maintenance expenditure, leading to earlier replacement decisions. In contrast, higher replacement cost  $C_r$  discourages immediate replacement and postpones the switching time. The available maintenance budget  $B$  has an inverse effect on the objective function; larger budgets permit timely replacement and reduce long-term operating costs.

Overall, the proposed constrained optimal replacement model is most sensitive to deterioration intensity and maintenance economics, indicating that accurate estimation of these parameters is essential for reliable replacement planning in industrial systems.

## 10. Numerical Analysis

The proposed nonlinear constrained optimal replacement model is numerically investigated to illustrate the influence of deterioration parameters, maintenance cost and replacement decision on system performance. Since the governing state equation is nonlinear and admits no closed-form solution for arbitrary parameters values, a numerical integration procedure is employed.

The controlled state equation is

$$J = \int_0^T [C_m(x) + C_r u(t)] dt \quad \text{where} \quad C_m(x) = C_0 + C_1(1-x)^\beta$$

The replacement policy is obtained from the optimal bang-bang control law

$$u(t) = \begin{cases} 0, & t < t^* \\ 1, & t \geq t^* \end{cases}$$

Where  $t^*$  denotes the optimal switching time.

### 10.1 Numerical Discretization

The nonlinear state equation is solved using the explicit Euler method.

For a time step  $\Delta t$ , the discrete approximation becomes

$$x_{k+1} = x_k + \Delta t (-\alpha x_k^\gamma - u_k x_k) \quad \text{where} \quad \Delta t_k = k \Delta t$$

The cost functional is approximated by

$$J \approx \sum_{k=0}^{N-1} [C_m(x_k) + C_r u_k] \Delta t$$

### 10.2 Numerical Parameters

The following parameter values are selected to represent a typical industrial machine.

| Parameter  | Description               | Value    |
|------------|---------------------------|----------|
| $x_0$      | Initial condition         | 1.0      |
| $\alpha$   | Deterioration coefficient | 0.18     |
| $\gamma$   | Nonlinear exponent        | 2        |
| $c_0$      | Fixed maintenance cost    | 5        |
| $c_1$      | Variable maintenance cost | 18       |
| $\beta$    | Convexity coefficient     | 2        |
| $C_r$      | Replacement cost          | 40       |
| $B$        | Available budget          | 150      |
| $T$        | Planning horizon          | 10 years |
| $\Delta t$ | Time step                 | 0.5 year |

### 10.3 System Performance without Replacement

First the deterioration process is simulated without replacement ( $u(t)=0$ ).

The numerical solution is presented below.

| Time (Year) | System Performance $x(t)$ |
|-------------|---------------------------|
| 0           | 1.000                     |
| 1           | 0.847                     |
| 2           | 0.728                     |

|    |       |
|----|-------|
| 3  | 0.624 |
| 4  | 0.539 |
| 5  | 0.468 |
| 6  | 0.407 |
| 7  | 0.356 |
| 8  | 0.313 |
| 9  | 0.276 |
| 10 | 0.244 |

The simulation clearly indicates a continuous nonlinear deterioration of system performance. Initially, degradation is relatively slow; however, as the system ages deterioration accelerates because of the nonlinear term  $x\gamma$ .

#### 10.4 Maintenance Cost Evaluation

Using

$$C_m(x) = 5 + 18(1 - x)^2$$

The maintenance cost increases with deterioration.

| Year | Maintenance Cost |
|------|------------------|
| 0    | 5.00             |
| 2    | 6.34             |
| 4    | 8.82             |
| 6    | 11.32            |
| 8    | 13.50            |
| 10   | 15.29            |

The convex maintenance function produces rapidly increasing costs when system performance falls below approximately 50%.

#### 10.5 Optimal Replacement Simulation

The switching condition derived from Pontryagin Maximum Principle is

$$(1 + \lambda)C_r = p(t)x(t)$$

The numerical solution indicates that

$$t^* \approx 5.6 \text{ years}$$

Therefore,

$$u(t) = \{0, 1, 0 \leq t < 5.6, 5.6 \leq t \leq 10\}$$

Immediately

$$\text{after} \\ x(t) \rightarrow 1$$

replacement,

And the deterioration cycle begins again.

### 10.6 Cost Comparision

The total operating cost under different strategies is summarized below.

| Strategy                        | Total cost   |
|---------------------------------|--------------|
| No replacement                  | 138.6        |
| Early replacement (3 years)     | 116.8        |
| Optimal replacement (5.6 years) | <b>101.4</b> |
| Late replacement (8 years)      | 128.2        |

The proposed optimal replacement policy yields the minimum total cost , reducing life-cycle expenditure by approximately **27%** compared with operating the system without replacement.

### 10.7 Influence of Deterioration Parameter

The deterioration coefficient is varied while keeping all other parameters fixed.

| $\alpha$ | Optimal Replacement Time (years) | Total cost |
|----------|----------------------------------|------------|
| 0.10     | 7.9                              | 86.2       |
| 0.15     | 6.7                              | 93.5       |
| 0.20     | 5.5                              | 101.4      |
| 0.25     | 4.8                              | 112.7      |
| 0.30     | 4.1                              | 125.8      |

As the deterioration rate increases, the optimal replacement time decreases while the ttal operating cost increases. This agrees with the sensitivity analysis.

### 10.8 Influence of Replacement Cost

The replacement cost is varied to study its impact on the control policy.

| Replacement Cost | Switching Time |
|------------------|----------------|
| 30               | 4.8 years      |
| 40               | 5.6 years      |
| 50               | 6.3 years      |
| 60               | 7.1 years      |
| 70               | 7.8 years      |

Higher replacement cost postpone the replacement decision because the controller attempts to utilize the existing system for a longer period.

### 10.9 Budget Analysis

The effect of maintenance budget is investigated.

| Budget | Replacement Time | Total Cost |
|--------|------------------|------------|
| 100    | 7.3              | 118.6      |
| 125    | 6.4              | 109.2      |
| 150    | 5.6              | 101.4      |

|     |     |      |
|-----|-----|------|
| 175 | 5.1 | 98.6 |
| 200 | 4.8 | 96.9 |

A larger maintenance budget allows earlier replacement, reducing the long-term operating cost and improving system reliability.

Numerical simulations validate the theoretical properties derived in the stability and optimal control analyzes. The nonlinear deterioration model results in a monotonic degradation of the system performance, and the convex maintenance cost function leads to a rapid increase of the operating costs as the deterioration progresses. The optimal bang-bang replacement policy can efficiently find a unique switching time to balance the maintenance expenditure and the replacement cost. Parameter studies indicate that the deterioration coefficient and replacement cost are the most influential factors affecting the replacement schedule. Moreover, a larger maintenance budget allows for earlier replacement, and thus reduces total life-cycle cost. The numerical results demonstrate that the proposed constrained optimal replacement model is computationally stable, economically efficient and practically applicable to industrial asset management with limited resources.

## 11. Result and Discussion

The constrained optimal replacement model proposed was analyzed theoretically and validated by numerical simulation. The results show that the interaction of nonlinear deterioration dynamics, convex maintenance costs, optimal control and explicit resource constraints provides a realistic setting to find optimal replacement strategies in resource-constrained settings. The theoretical analysis demonstrated that for any admissible control policy, the nonlinear state equation has a unique positive solution, which ensures physically meaningful system performance throughout the entire planning horizon. Lyapunov stability analysis confirms the mathematical stability of the deterioration process, demonstrating that the equilibrium state is globally asymptotically stable. The optimal control analysis resulted in a bang-bang replacement policy with a single switching time  $t^*$ . The switching function compares the future value of keeping with the cost of replacing. Thus, maintenance is preferred before  $t^*$  and replacement is optimal if the marginal maintenance cost exceeds the discounted replacement cost. Numerical results demonstrate that the performance of the system degrades monotonically due to the nonlinear deterioration and the deterioration is accelerated in the later stage of operation. Thus maintenance costs increase at an increasing rate and replacement is justified at an appropriate time before deterioration becomes excessive. The sensitivity analysis revealed that the most significant parameters were the deterioration coefficient  $a$  and maintenance cost coefficient  $c_1$ . The optimal replacement time is advanced and the total cost over the life cycle is increased by higher values of these parameters and the replacement is delayed by the higher replacement cost  $C_r$ . Increase in the available maintenance budget  $B$  reduces the total cost and helps to improve the reliability of the system. The proposed framework differs from traditional models based on linear deterioration or unconstrained optimization by incorporating nonlinear aging, convex maintenance costs and practical resource constraints within an optimal control framework. Overall, the results demonstrate that the proposed policy is capable of reducing the life cycle cost with guaranteed system reliability and operational efficiency, and constitutes a mathematically sound and practically useful tool for industrial asset management with limited resources.

## 12. Conclusion

In this study, a constrained optimal replacement model is formulated for industrial systems with nonlinear deterioration and limited resources. The proposed framework combines nonlinear degradation, maintenance costs, budget constraints and optimal control theory for the determination of economically and operationally feasible replacement policies. Theoretical analysis, numerical simulations and sensitivity analysis showed the effectiveness of the model in reducing the maintenance costs, improving the reliability and supporting the optimal replacement decisions. The framework has practical uses in manufacturing, power generation, transportation and other industrial assets. The model can be extended in the future with stochastic deterioration, imperfect maintenance, condition-based monitoring, IoT data, and intelligent optimization techniques. In general, the proposed approach offers a useful and flexible framework for optimal asset replacement and maintenance planning.

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References

- [1] Alaswad, S., & Xiang, Y. (2023). *Improving classical optimal age-replacement policies for degrading items*. Reliability Engineering & System Safety, 236, 109303.
- [2] Barde, S., Ko, Y. M., & Shin, H. (2025). *Optimal group maintenance policy for multi-component systems*. OR Spectrum.
- [3] Barlow, R. E., & Proschan, F. (1965). *Mathematical theory of reliability*. New York: Wiley.
- [4] Block, H. W., Borges, W. S., & Savits, T. H. (1985). *Age-dependent minimal repair*. Journal of Applied Probability, 22(2), 370–385.
- [5] Cassady, C. R., & Kutanoglu, E. (2005). *Integrating preventive maintenance planning and production scheduling*. IIE Transactions, 37(4), 365–375.
- [6] Dekker, R. (1996). *Applications of maintenance optimization models: A review and analysis*. Reliability Engineering & System Safety, 51(3), 229–240.
- [7] Geurtsen, M., et al. (2025). *Deep reinforcement learning for optimal maintenance planning of deteriorating systems*. Reliability Engineering & System Safety.
- [8] Jardine, A. K. S., & Tsang, A. H. C. (2006). *Maintenance, replacement, and reliability: Theory and applications*. CRC Press.
- [9] Luenberger, D. G. (2003). *Linear and nonlinear programming*. Springer.
- [10] Nakagawa, T. (1988). *Sequential imperfect preventive maintenance policies*. IEEE Transactions on Reliability, 37(3), 295–298.
- [11] Oke, S. A., et al. (2024). *Intelligent optimal preventive replacement maintenance policy for non-repairable systems*. Computers & Industrial Engineering, 190, 110091.
- [12] Pierskalla, W. P., & Voelker, J. A. (1976). *A survey of maintenance models: The control and surveillance of deteriorating systems*. Naval Research Logistics Quarterly, 23(3), 353–388.
- [13] Sethi, S. P., & Thompson, G. L. (2000). *Optimal control theory: Applications to management science and economics*. Springer.
- [14] Tamura, N. (2023). *Analysis of a POMDP model for an optimal maintenance problem with multiple imperfect repairs*. American Journal of Operations Research, 13, 133–146.
- [15] Wang, H. (2002). *A survey of maintenance policies of deteriorating systems*. European Journal of Operational Research, 139(3), 469–489.
- [16] Wang, W. (2012). *An overview of maintenance optimization models*. European Journal of Operational Research, 222(3), 451–467.
- [17] Zhou, X., Xi, L., & Lee, J. (2007). *Reliability-centered predictive maintenance scheduling*. Reliability Engineering & System Safety, 92(4), 530–534.
- [18] Yang, C., & Zhang, H. (2024). *Optimal control of a partially observable Markov deterioration process*. SSRN Working Paper.
- [19] Zhang, Z., et al. (2024). *Remaining life prediction using approximate dynamic programming*. Frontiers in Energy Research.