

Analysis of Low Free Vibration Modes Large Branched Gear Mechanisms Using Dqds Algorithm

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Abstract- The main impartial of this paper is to propose a new empirical process for computing the most dominant vibration modes of a large branched gearing mechanism in a lathe. This method utilizes the algorithm, which is derived from the LR algorithm and is particularly effective for finding the eigenvalues of large matrices. The gearing mechanism in question consists of multiple gears, resulting in numerous dynamic parameters that need consideration. To simplify the dynamic modelling process, the system is divided into subsystems. However, even the stiffness matrices derived for these subsystems are quite large, making it complex and time-consuming to compute the vibration modes from these matrices.

To address these difficulties, the algorithm is employed to calculate the vibration modes of each subsystem. Typically, most systems operate within the mid-frequency range. As a result, this empirical method allows for the effective exclusion of less dominant vibration modes associated with high-frequency ranges. This simplifies the system model and reduces computational complexity significantly. This reduction method offers several advantages, including a simplified system model and reduced computational demands.

Keywords: Lanczos Reduction (LR), the large branched system, the algorithm, a new empirical mode reduction method, the working frequency range, the most dominant vibration modes.

1. Introduction

In this way, a new mode reduction process is applied to a large branched (Multi Input Multi Output) gearing mechanism of a lathe. The system is very large, therefore it is very difficult to compute the free vibration modes from the dense stiffness matrix. Hence, in order to simplify the dynamical modelling the system is divided into the subsystems and

each subsystem is considered as a independent system. The number of the subsystems are proportional to the number of the branches on the system. However, the stiffness matrix of each subsystem is still dense and the computation of the free vibration modes from this matrix is complicated and time consuming. For this reason, the algorithm is used to compute the free vibration modes of the dense stiffness matrices of the subsystems. The mode reduction method that is proposed in this paper is a new empirical method and it is very suitable for the large systems in order to reduce the computation time and efforts. The most dominant vibration modes of the each subsystem are computed separately by using the empirical mode reduction method. Then, the most dominant modes of the whole system are computed by bringing together the most dominant vibration modes of each subsystem (generally, they are the first three modes). This is a rather acceptable method in practice due to the fact that generally most systems work in the mid-frequency range. For this reason, it is not necessary to consider vibration modes which are inside of the high-frequency range owing to the fact that they will not effect the system output response.

The algorithm was used to compute the free vibration modes (the eigenvalues) of the dense symmetric matrix by B.N. Parlett [1]. The algorithm was derived from LR algorithm [2] and it is very effective for computing the eigenvalues of large matrices. The most algorithms for computing eigenvalues of large matrices many times generate small auxiliary matrices whose Eigen systems need to be computed as a subtask. The mass common algorithms for this subtask are the GR algorithm and its variant [3].

The family of Generalized Rutishauser (GR) algorithms, which incorporates the LR and Quotient-Remainder (QR) algorithms are iterative ways that move a matrix toward upper-triangular form via similarity transformations. The subject was born with Rutishauser's quotient-difference algorithm [4]. In this work, the method was used for calculating the poles of meromorphic function. The convergence theory of GR algorithms was discussed by D.S. Watkins and L. Elsner [5] and summarized in [6]. V.N. Kublanovskaya [7] was used the QR algorithm for calculating the eigenvalues and eigenvectors of small matrices. J. G. F. Francis [8] introduced the double-shift implicit QR algorithm that is in widespread use today only a few details are removed.

In [9] a family of GR algorithms was introduced, and a common convergence theorem for unshifted GR algorithms were proved. D.S. Watkins and L. Elsner [5] introduced a more common family of GR algorithms and proved general convergence theorems for shifted GR algorithms. The unshifted QR algorithm convergence was proved using matrices, not determinants by Wilkinson [10]. A shifted QR algorithm convergence was also proved on symmetric, tridiagonal matrices by Wilkinson [11] and B.N. Parlett [12].

Nowadays, Matlab is the most popular software package for matrix computation. It has four QR- based computational kernels, which are a few refinements removed from codes in the public-domain software packages EISPACK[13] and LINPACK [14]. LAPACK is another software package, which is in common use as well and it is newer in public-domain [15].

The eigenvectors of a symmetric tridiagonal matrix were computed by using MRRR (Multiple Relatively Robust Representation) algorithm [16]. In this work, each eigenvectors was computed with acclaim to the appropriate depiction. However, MRRR subset computations for the symmetric tridiagonal eigenproblem are more intricate than inverse iteration and the method sometimes cannot pass over the unfavorable part of the spectrum [17].

The LR algorithm was presented by Rutishauser in 1957, see [18], along with a proof of its surprising convergence property. For simplicity, all shifts were taken to be zero. Hence, if $\mathbf{A}$ has eigenvalues with distinct absolute values and if the factorization does not fail then, very slowly, the q -values tend to the eigenvalues in monotonic decreasing order, i.e., q_n tends to the eigenvalue closest to zero.

The advantages of the dqds algorithm over qds is that it preserves high mixed relative stability even in the presence of element growth. Small end-figure changes to the input $\mathbf{L}$, $\mathbf{U}$ and to the output $\hat{\mathbf{L}}$, $\hat{\mathbf{U}}$ give an exact transformation and this feature permits all eigenvalues to be computed to high relative accuracy in the positive case. When the original $\mathbf{A}$ comes from a positive definite symmetric matrix $\mathbf{T}$, via $\mathbf{A} = \Delta \mathbf{T} \Delta^{-1}$, then all q 's and e 's will be positive. This shift must be chosen carefully and details can be found in [2] and [19].

Apart from its use in computing eigenvalues; the dqds transform plays a role in computing ‘twisted’ factorizations of tridiagonals that are needed in the course of computing eigenvectors, see [20, 21]. Rutishauser implemented the LR transform so that $\hat{A}$ overwrote A with no explicit reference to L , U and the e -values tend to zero. The symmetric eigenvalue problem for dense matrices was studied by Parlett [1]. In this paper, it was proved that LDU is also preferable for eigenvalue problem, particularly in the symmetric tridiagonal matrix. Also, it was claimed that in the symmetric case T must be replaced or translated by its triangular factors LDL^T when some of eigenvalues were tightly clustered.

2. The Free Vibration Modes Analyses of the Large Branched System

A large branched rotational system can be described by an inertial system, in which elements of each vibration mode are not represented explicitly. Since each vibration mode is affected by some motion modes, it is hard to suppress vibrations independently in the inertial system. The free vibration of the system with n degrees-of-freedom (d.o.f.) is written in the physical space by the following second order differential equation

$$J\ddot{\theta}(t) + K\theta(t) = 0 \quad (1)$$

where J and K are the inertia and stiffness matrices, respectively and $\theta(t)$ is the angular displacement of the system. Both the inertia and stiffness matrices are symmetric and positive definite. From (1), the below equalities can be written

$$\theta(t) = Ae^{i\omega t}, \dot{\theta}(t) = i\omega Ae^{i\omega t} \text{ and } \ddot{\theta}(t) = -\omega^2 Ae^{i\omega t}. \quad (2)$$

Where $\dot{\theta} = [\dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3 \ \dots \ \dot{\theta}_n]^T$, $\theta = [\theta_1 \ \theta_2 \ \theta_3 \ \dots \ \theta_n]^T$ and n denotes the number of degrees of freedom of each subsystem. From (1) and (2) the following equation is written as

$$e^{i\omega t} (KA - J\omega^2 A) = 0 \quad (3)$$

where $e^{i\omega t} = \cos \omega t + i \sin \omega t \neq 0$; then (3) is rewritten as

$$(K - J\omega^2)A = 0 \quad (4)$$

where, ω is the natural frequency and A is the amplitude matrix of the system, $A = [A_1, A_2, \dots, A_n]^T$. In order to solve (4), the following inequality must be considered, $A \neq 0$. Hence (4) becomes

$$\begin{aligned} \det(K - \omega^2 J) &= 0 \\ \det(K - \lambda J) &= 0 \end{aligned} \quad (5)$$

where $\lambda = \text{diag}(\lambda_i) = \text{diag}(\omega_i^2)$ and $i = 1, 2, 3, \dots, n$. The solution of equation (5) will give us the roots of a n -th degree of freedom differential equation which is function of λ_i . The roots of equation (5) are corresponding to the eigenvalues (the free vibration modes) of the system. The corresponding natural frequencies of the vibration system are computed from $\omega_i = \sqrt{\lambda_i}$. Then, the amplitude matrix of each subsystem, A_i is written as below

$$A_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \quad (6)$$

If a system consists of n -branches then the whole amplitude matrix of the gearing mechanism can be written as

$$\Phi = [A_1, A_2, A_3, \dots, A_n]^T =$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$
(7)

where Φ is the modal matrix [1] that satisfies

$$\theta(t) = \Phi \xi(t),$$
(8)

where $\xi(t)$ is an n dimensional modal coordinate system that is written as follows

$$\xi(t) = [\xi_1(t) \quad \xi_2(t) \quad \xi_3(t) \quad \cdots \quad \xi_n(t)]^T$$
(9)

Substituting (8) into (1) and multiplying Φ^T on both sides from the left yields

$$\Phi^T J \Phi \ddot{\xi}(t) + \Phi^T K \Phi \xi(t) = 0$$
(10)

Generally, the coefficient matrices $\Phi^T J \Phi$ and $\Phi^T K \Phi$ are diagonalized due to the general orthogonality of the modal matrix.

$$J = \begin{bmatrix} j_1 & 0 & 0 & \cdots & 0 \\ 0 & j_2 & 0 & \cdots & 0 \\ 0 & 0 & j_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & j_n \end{bmatrix}$$
(11)

The modal matrix relates the motion and vibration modes can be easily estimated if physical parameters such as the mass and stiffness matrices of the system are known. However, in practice not all physical parameters are known a priori. If this is the case, the mass and stiffness matrices can be estimated from the frequency function or the impulse response function of the vibration system.

$$K = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -k_3 & (k_3 + k_4) & -k_4 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -k_4 & (k_4 + k_5) & -k_5 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -k_{n-3} & (k_{n-3} + k_{n-2}) & -k_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -k_{n-2} & (k_{n-2} + k_{n-1}) & -k_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & (k_{n-1} + k_n) & -k_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$
(12)

3. Dividing the System into the Subsystems

In order to simplify the dynamical modelling of the large branched gearing mechanism the system is divided into the subsystems and the mass and stiffness matrices of each subsystem are computed separately. The number of subsystems are proportional to the number of branches which the systems have. For this reason, two branches looped gearing mechanism which is shown in Fig. 1 is divided into two subsystems.

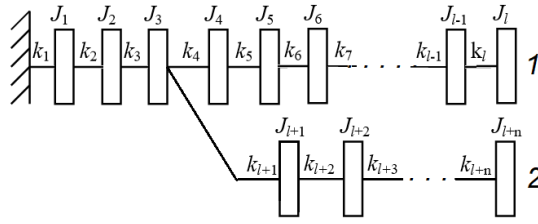


Fig. 1: Two branches dynamic model

For two branches gearing mechanism the stiffness matrix of the whole system is written as below

$$K = \begin{bmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{bmatrix} \quad (13)$$

where $\dim(K_1) = lxl$ and $\dim(K_2) = l + nxl + n$, K_1, K_2 are the triadiagonal stiffness matrices of the first and second subsystems. K_{12}, K_{21} are the interaction stiffness matrices which are relating the first subsystem to the second subsystem and the second subsystem to the first subsystem respectively. The terms in (13) are written as

$$K_1 = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & 0 & 0 & \cdots & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & 0 & 0 & \cdots & 0 \\ 0 & -k_3 & (k_3 + k_4) & -k_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -k_{l-2} & (k_{l-2} + k_{l-1}) & -k_{l-1} & 0 \\ 0 & 0 & \cdots & 0 & -k_{l-1} & (k_{l-1} + k_l) & -k_l \\ 0 & 0 & 0 & \cdots & 0 & -k_l & k_l \end{bmatrix} \quad (14)$$

$$K_2 = \begin{bmatrix} (k_{l+1} + k_{l+2}) & -k_{l+2} & 0 & 0 & 0 & \cdots & 0 \\ -k_{l+2} & (k_{l+2} + k_{l+3}) & -k_{l+3} & 0 & 0 & \cdots & 0 \\ 0 & -k_{l+3} & (k_{l+3} + k_{l+4}) & -k_{l+4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -k_{l+n-2} & (k_{l+n-2} + k_{l+n-1}) & -k_{l+n-1} & 0 \\ 0 & 0 & \cdots & 0 & -k_{l+n-1} & (k_{l+n-1} + k_{l+n}) & -k_{l+n} \\ 0 & 0 & 0 & \cdots & 0 & -k_{l+n} & k_{l+n} \end{bmatrix} \quad (15)$$

$$K_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ -k_{l+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{lxl} \quad (16)$$

$$K_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ -k_{l+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{lxl} \quad (17)$$

In order to reduce the computation time and efforts two branches of the system are analysed separately and it is considered that there is no interaction between them. If these two subsystems are independent of each other then, the interaction stiffness matrices K_{12} and K_{21} can be ignored. Thus, the terms K_{12} and K_{21} in (13) are set to zero and the stiffness matrix of two branches system is rewritten as follows

$$K_0 = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \quad (18)$$

where K_0 is representing the stiffness matrix of two branches system which has no interaction between the first and second subsystems of it. The natural frequencies and spectral matrices of two branches dynamic system are computed from (5) by using the dqds algorithm which is explained in detail in Section 4. Hence, the spectral matrices of the first and second subsystems are

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_l \end{bmatrix} = \text{diag}(\lambda_i) \quad (19)$$

$$\Lambda_2 = \begin{bmatrix} \lambda_{l+1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{l+2} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{l+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{l+n} \end{bmatrix} = \text{diag}(\lambda_{l+i}) \quad (20)$$

where $j = 1, 2, 3, \dots, l$ and $i = 1, 2, 3, \dots, n$. Thus, the modal matrices of the first and second branches are

$$\begin{aligned} \Phi_1 &= (A_1, A_2, A_3, \dots, A_l)^T = \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1l} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2l} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l1} & a_{l2} & a_{l3} & \cdots & a_{ll} \end{bmatrix} \end{aligned} \quad (21)$$

$$\begin{aligned} \Phi_2 &= (A_{l+1}, A_{l+2}, A_{l+3}, \dots, A_{l+n}) = \\ &= \begin{bmatrix} a_{l+1l+1} & a_{l+1l+2} & a_{l+1l+3} & \cdots & a_{l+1l+n} \\ a_{l+2l+1} & a_{l+2l+2} & a_{l+2l+3} & \cdots & a_{l+2l+n} \\ a_{l+3l+1} & a_{l+3l+2} & a_{l+3l+3} & \cdots & a_{l+3l+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l+n1} & a_{l+n2} & a_{l+n3} & \cdots & a_{l+n1+n} \end{bmatrix} \end{aligned} \quad (22)$$

$$\Phi_1^T K_1 \Phi_1 = \Lambda_1 \quad (23)$$

$$\Phi_2^T K_2 \Phi_2 = \Lambda_2 \quad (24)$$

The modal matrices of the first and second independent subsystems are written in a matrix form as

$$\begin{bmatrix} \Phi_1^T & 0 \\ 0 & \Phi_2^T \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \quad (25)$$

Equation (25) is rewritten as

$$\Phi_0^T K_0 \Phi_0 = \Lambda_0 \quad (26)$$

where Φ_0 and Λ_0 are representing the modal and spectral matrices respectively which are calculated summing of the dynamic characteristics of the first and second independent subsystems. If these two subsystems are not independent of each other then, the above expression may rearranged as follows

$$\begin{aligned}\Phi_0^T K_0 \Phi_0 &= \begin{bmatrix} \Phi_1^T & 0 \\ 0 & \Phi_2^T \end{bmatrix} \begin{bmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{bmatrix} \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \\ &= \begin{bmatrix} \Phi_1^T K_1 \Phi_1 & \Phi_1^T K_{12} \Phi_2 \\ \Phi_2^T K_{21} \Phi_1 & \Phi_2^T K_2 \Phi_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{21} & \Lambda_2 \end{bmatrix}\end{aligned}\quad (27)$$

In the above equation Λ_{12} is representing the connection matrix which is relating the first subsystem to the second subsystem.

Similarly, Λ_{21} is also the connection matrix which is relating the second subsystem to the first subsystem. Both Λ_{12} and Λ_{21} matrices are representing the normal modes between the first and second subsystems of the branched gearing mechanism. As it is mentioned before these two

subsystems assumed not affecting each other performances, in order to simplify the dynamic modelling therefore, the connection matrices are set to zero. Three branches gearing mechanisms are shown in Fig. 2a, b.

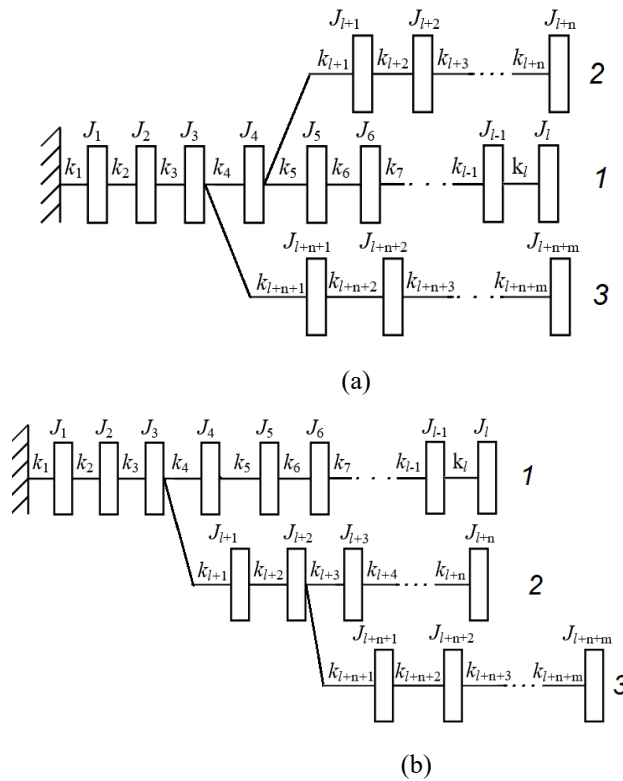


Fig. 2: Three branches gearing mechanism

The stiffness matrix of three branches gearing mechanism that is shown in Fig. 2a is written as below

$$K_{(a)} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{bmatrix}\quad (28)$$

In the above stiffness matrix, $K_{(a)}$ the terms K_{23} and K_{32} are zero owing to the fact that there is not direct connection between the second and third links. In a similar manner, the stiffness matrix of the system which is shown in Fig. 2b is written as follows

$$K_{(b)} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix} \quad (29)$$

Similarly, there is not direct connection between the first and third links; therefore in $K_{(b)}$, the values of K_{13} and K_{31} are zero. In practice, the subsystems of the branched gearing systems are not independent of each other as shown in Fig. 1 and Fig. 2. The subsystems work together and they affect each others performances. However, in this work in order to simplify the dynamical modelling and reduce the computation time and efforts each subsystem is modelled as an independent system.

4. The dqds Algorithm

GR algorithms [i] are iterative methods which were used to compute the eigenvalues of large matrices. The algorithm begin with a matrix A_o similar to A (e.g. $A_o = A$) and produce a sequence (A_m) of similar matrices. All GR algorithms have the following form. Given the iterate

A_m , the next iterate A_{m+1} is produced as follows. The first a spectral transformation function f_m is somehow chosen. At this point the only requirement on f_m is that the matrix $f_m(A_m)$ be well defined. Thus f_m could be a polynomial, rational function.

If T is a symmetric matrix there is a variant of the well-known bisection algorithm, see [ii], that may be used. This method is effective for refining eigenvalues already known to good accuracy but is slow as a general tool. There is a much faster method, discovered by Fernando and Parlett, see [2], that computes the eigenvalues of LL' . If $A = LU$ then the LR transform of A is $\hat{A} = UL$. When shifts are incorporated we obtain LR(s):

factor $A - sI = LU$

form $\hat{A} = UL + sI$

This may be accomplished by the qds algorithm that was discovered by Rutishauser, see [4], he was discovered that qds was equivalent to LR in exact arithmetic. However this is not the most accurate implementation. There is a differential form of qds(s) that was discovered by K.V. Fernando and Parlett, see [2] and the method uses A.L. Cholesky factorization. An examination of both the qds(s) and dqds(s) algorithms show that the shift is not restored. Thus all eigenvalues have been reduced by s . In practice, with the dqds algorithm, the program checks when e_n and q_n are both negligible and then records s as an eigenvalue and reduces the order n by 1. Let T be an $n \times n$ real matrix whose eigenvalues we seek.

$$T = \begin{bmatrix} d_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_2^2 & d_2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & b_3^2 & d_3 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & b_4^2 & d_4 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & b_{n-2}^2 & d_{n-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & b_{n-1}^2 & d_{n-1} & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & b_n^2 & d_n \end{bmatrix}_{n \times n} = A \quad (30)$$

$$A - sI = LU \quad (31)$$

$$\text{Where } L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ e_2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & e_3 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & e_4 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & e_5 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_n & 1 \end{bmatrix}_{n \times n},$$

$$U = \begin{bmatrix} q_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & q_3 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & q_4 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & q_5 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & q_n \end{bmatrix}_{n \times n} \quad (32)$$

Equation (31) is rewritten as

$$A - sI = LU \Rightarrow A = LU + sI \quad (33)$$

$$\begin{aligned} &= \begin{bmatrix} d_1 - s & 1 & 0 & 0 & \cdots & 0 \\ b_2^2 & d_2 - s & 1 & 0 & \vdots & 0 \\ 0 & b_3^2 & d_3 - s & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & b_{n-2}^2 & d_{n-1} - s & 1 \\ 0 & 0 & \cdots & 0 & b_{n-1}^2 & d_n - s \end{bmatrix} = \\ &= \begin{bmatrix} q_1 & 1 & 0 & 0 & \cdots & 0 \\ e_2 q_1 & e_2 + q_2 & 1 & 0 & \cdots & 0 \\ 0 & e_3 q_2 & e_2 + q_3 & 1 & & \\ 0 & 0 & e_4 q_3 & e_4 + q_4 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & e_n q_{n-1} & e_n + q_n \end{bmatrix} \end{aligned} \quad (34)$$

From the above equation below equalities can be written

$$q_1 = d_1 - s, \quad e_i = \frac{b_i^2}{q_{i-1}}, \quad q_i = d_i - e_i - s \quad (35)$$

where $i = 2, 3, 4, \dots, n$ and

$$A = \begin{bmatrix} q_1 + s & 1 & 0 & 0 & \cdots & 0 \\ e_2 q_1 & e_2 + q_2 + s & 1 & 0 & \cdots & 0 \\ 0 & e_3 q_2 & e_3 + q_3 + s & 1 & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e_{n-1} q_{n-2} & e_{n-1} + q_{n-1} + s & 1 \\ 0 & 0 & \cdots & 0 & e_n q_{n-1} & e_n + q_n + s \end{bmatrix} \quad (36)$$

Rutishauser's qds transform (qd with shifts) [4], [18] and the dqds transform each with shift s map A into $\hat{A}$ where the associated bidiagonals $\hat{L}$, $\hat{U}$ satisfy

$$\begin{aligned}\hat{A} - sI &= \hat{L}\hat{U} - sI = UL \\ \hat{A} &= UL + sI\end{aligned}\quad (37)$$

Application of (37) to (33) yields

$$\begin{aligned}\hat{A} &= (LU + sI) = UL + sI \\ \hat{A} &= \hat{L}\hat{U} = UL + sI\end{aligned}\quad (38)$$

$$\hat{A} = \begin{bmatrix} q'_1 + s & 1 & 0 & 0 & \cdots & 0 \\ e'_2 q'_1 & e'_2 + q'_2 + s & 1 & 0 & \cdots & 0 \\ 0 & e'_3 q'_2 & e'_3 + q'_3 + s & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e'_{n-1} q'_{n-2} & e'_{n-1} + q'_{n-1} + s & 1 \\ 0 & 0 & \cdots & 0 & e'_n q'_{n-1} & e'_n + q'_n + s \end{bmatrix} \quad (39)$$

and

$$UL + sI = \begin{bmatrix} q_1 + e_2 + s & 1 & 0 & 0 & \cdots & 0 \\ q_2 e_2 & q_2 + e_3 + s & 1 & 0 & \cdots & 0 \\ 0 & q_3 e_3 & q_3 + e_4 + s & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & q_{n-1} e_{n-1} & q_{n-1} + e_n + s & 1 \\ 0 & 0 & \cdots & 0 & e_n q_n & q_n + s \end{bmatrix} \quad (40)$$

From (39) and (40) the following equalities are written

$$\begin{aligned}q'_1 &= q_1 + e_2; \quad q'_i = q_i - e'_i + e_{i+1}; \quad e'_i = \frac{e_i q_i}{q'_{i-1}}; \\ d'_i &= q_{i-1} + e_i; \quad d'_n = q_n\end{aligned}\quad (41)$$

together with

$$\begin{aligned}e'_i q'_{i-1} &= e_i q_i \Rightarrow \frac{e_i}{q'_{i-1}} = \frac{e'_i}{q_i} \\ q'_i + e'_i &= q_i + e_{i+1} \Rightarrow q'_i - e_{i+1} = q_i - e'_i\end{aligned}\quad (42)$$

However, doing all these computation by hand is rather difficult, complicated and time consuming. For these reasons a program can be written and the natural frequencies and vibration modes of a large branched system may be computed rather easily. The program will be written based on the dqds algorithm equations, as presented in (30)-(42).

5. The Reduction of the Vibration Modes

Finding the eigenvalues of an $n \times n$ matrix is equivalent to solving for the roots of an n -th degree polynomials. For a mechanical system the eigenvalues represent the normal modes of vibration. However, if $n \geq 3$, the characteristic polynomial may be difficult to solve. The reduction procedure for the modal and spectral matrices are described graphically in Figs. 3 and 5 respectively.

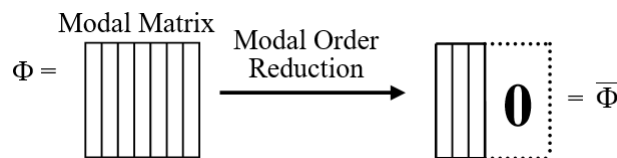


Fig. 3: The reduction of the modal matrix

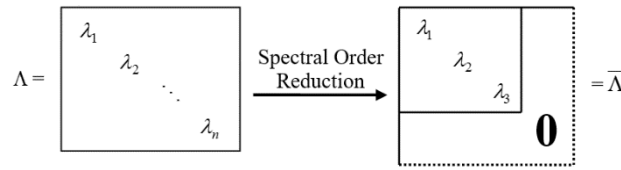


Fig. 4: The reduction of the vibration modes

According to Fig. 4 and the following inequality, $\omega_k \gg \omega_p$ the angular velocity ω_k is ignored due to the fact that it is extremely bigger than the nominal working velocity, ω_p . As a result, the corresponding vibration mode and angular velocity ($\lambda_k = \omega_k^2$) are also ignored. Let, the first and second branches of the system have 40 and 20 degrees of freedom respectively. Then, the whole system will have 60 degrees of freedom therefore the spectral matrix will be 60x60. If the system dimensions are reduced 10 times than the reduced order spectral matrix of the whole system will be 6x6. Hence, after the reduction procedure the spectral matrix of the system, will have much smaller dimensions as shown schematically in

$$\bar{\Phi}^T K \bar{\Phi} = \begin{matrix} \bar{\Phi}^T \\ \hline \hline \hline \end{matrix} \times \begin{matrix} K \\ \hline \hline \hline \end{matrix} \times \begin{matrix} \bar{\Phi} \\ \hline \hline \hline \end{matrix} = \begin{matrix} \bar{\Lambda}_1 & \bar{\Lambda}_{12} \\ \hline \bar{\Lambda}_{21} & \bar{\Lambda}_2 \end{matrix} = \bar{\Lambda}$$

Fig. 5: The reduction of the spectral matrix

The strategy of the proposed method is explained as below:

1. The branched system is divided into the subsystems as shown in Figs. 1 and 2.
2. Each subsystem is analysed by the dqds algorithm in order to compute the most dominant vibration modes.
3. For two branches system all the stiffness matrices of the system are reduced according to the reduced spectral matrix which is shown in Fig. 5. Then, the reduced order spectral matrix that is representing the whole system is computed and written in matrix form. Hence, the diagonal elements of the spectral matrix are generated from the spectral analyses of the subsystems. However, the off-diagonal elements are computed by using the below equation

$$\bar{\Lambda}_{12} = \bar{\Phi}_1^T K_{12} \bar{\Phi}_2 \quad (43)$$

From (43) $\bar{\Lambda}_{21}$ is computed as

$$\bar{\Lambda}_{21} = \bar{\Lambda}_{12}^T \quad (44)$$

Similarly, the spectral matrix of three branches gearing mechanism (Fig. 2a, b) is written as follows

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_1 & \bar{\Lambda}_{12} & \bar{\Lambda}_{13} \\ \bar{\Lambda}_{21} & \bar{\Lambda}_2 & \bar{\Lambda}_{23} \\ \bar{\Lambda}_{31} & \bar{\Lambda}_{32} & \bar{\Lambda}_3 \end{bmatrix} \quad (45)$$

The off-diagonal elements $\bar{\Lambda}_{12}$ and $\bar{\Lambda}_{21}$ are computed from (43) and (44) respectively, then the rest of other off-diagonal elements of the matrix are derived as below

$$\begin{aligned} \bar{\Lambda}_{13} &= \bar{\Phi}_1^T K_{13} \bar{\Phi}_3, & \bar{\Lambda}_{31} &= \bar{\Lambda}_{13}^T \\ \bar{\Lambda}_{23} &= \bar{\Phi}_2^T K_{23} \bar{\Phi}_3, & \bar{\Lambda}_{32} &= \bar{\Lambda}_{23}^T \end{aligned} \quad (46)$$

If a system has n -branches then, the above equations can be generalized as follows

$$\bar{\Lambda}_{gh} = \bar{\Phi}_g^T K_{gh} \bar{\Phi}_h, \quad \bar{\Lambda}_{hg} = \bar{\Lambda}_{gh}^T \quad (47)$$

where $i = 1, 2, \dots, n-1$ and $j = i+1, i+2, \dots, n$ and the spectral matrix is

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_1 & \bar{\Lambda}_{12} & \cdots & \bar{\Lambda}_{1n} \\ \bar{\Lambda}_{21} & \bar{\Lambda}_2 & \cdots & \bar{\Lambda}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Lambda}_{n1} & \bar{\Lambda}_{n2} & \cdots & \bar{\Lambda}_{nn} \end{bmatrix}_{n \times n} \quad (48)$$

The diagonal elements $\bar{\Lambda}_1, \bar{\Lambda}_2, \dots, \bar{\Lambda}_k$ are computed from the spectral analysis of the subsystems.

4. Generally, the reduced spectral matrix has smaller dimension than the original matrix of the system. If a system consists of n -branches then, the spectral matrix may be written in (48). In this matrix, in order to compute all the elements ($\bar{\Lambda}_{ij}$), it is necessary to compute the diagonal elements in a first place. Then, it will be enough to compute the upper or lower triangular elements due to the fact that the matrix is symmetric.

5. The natural frequencies of the system are computed from the reduced spectral matrix as follows

$$\Phi = \bar{\Phi} W \quad (49)$$

where, $\bar{\Phi} = \bar{\Phi}_1 \oplus \bar{\Phi}_2 \oplus \dots \oplus \bar{\Phi}_k$ is the straight addition of the modal forms of the subsystems and W is the natural frequencies of the reduced system.

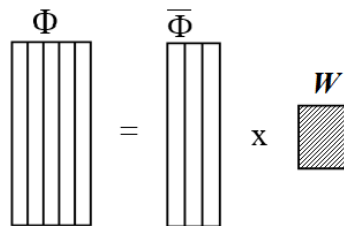


Fig. 6: The reduced free vibration forms

Equation (49) is described schematically in Fig. 6. The above equality is rewritten as

$$\begin{aligned} \Phi &= (\bar{\Phi}_1 \oplus \bar{\Phi}_2 \oplus \bar{\Phi}_3 \oplus \dots \oplus \bar{\Phi}_k) \times W \\ \Phi &= \Phi_1 + \Phi_2 + \Phi_3 + \dots + \Phi_k \end{aligned} \quad (50)$$

Thus, the reduced modal forms and the natural frequencies of the system may be computed by using the above equalities effectively.

6. Discussions

The characteristic forms, λ_i (square of the angular velocities) of the gearing mechanisms increase as double of the form numbers. Generally, a gearing mechanism has one or two dominant free vibration modes in the working frequency range. Similarly, each subsystem of the branched gearing mechanism has one or two free dominant vibration modes in their working frequency range.

Assume that the nominal angular velocity of a gearing mechanism is ω_c . Normally, the first dominant vibration mode which corresponds to the first dominant angular velocity, ω_1 is slightly bigger than the nominal angular velocity of the gearing mechanism, $\omega_1 \gtrsim \omega_c$ then, the following angular velocities, ω_i will be far more bigger than the nominal angular velocity, $\omega_i \gg \omega_c$. From these inequalities the below equations are written

$$\begin{aligned} \lambda_i &= \omega_i^2 - i^2 \omega_c^2 \\ \lambda_i &= i^2 \lambda_c \end{aligned} \quad (51)$$

where i is the reduced mode number, $i=1, 2, 3, \dots, k$ and λ_c is the characteristic vibration mode of the gearing mechanism which corresponds to the nominal angular velocity, ω_c .

In the case of two branches gearing mechanism the stiffness matrix of the whole system is formed according to the vibration forms of the first and second branches as described in (27). In this equation, diagonal elements, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are representing the reduced vibration forms of the first and second subsystems respectively. The off-diagonal elements $\bar{\lambda}_{12}$ and $\bar{\lambda}_{21}$ consist of elastic constants elements which are relating the first subsystem to the second subsystem and the second subsystem to the first subsystem respectively. As a result of these interactions the modes of the subsystems will apply dynamic forces to each other which are computed as follows

$$f_{i1} \cong \frac{\lambda_{i1}}{\lambda_i^{(1)} - \lambda_i^{(2)}}, f_{i1} = f_{i1} \quad (52)$$

where f_{i1} and λ_{i1} are representing the interactive dynamic force and the vibration modes of the off-diagonal elements $\bar{\lambda}_{i1}$ respectively. In the above equation $\lambda_i^{(1)}$ is representing the vibration modes (eigenvalue) of the first subsystem and $\lambda_i^{(2)}$ is representing the modes of the second subsystem of two branches looped gearing mechanism. The interactive dynamic force, f_{i1} will be maximum for the smaller vibration modes (the most dominant vibration modes). Certainly, it will be minimum for the bigger vibration modes (the less dominant vibration modes) of the subsystems. The meaning of the above equation is in order to analyse the system accurately the spectrum matrix may be rearranged according to the opposite interactive forces which are activated by the subsystems.

The dynamical modelling of two branches gearing mechanism is simplified by setting the off-diagonal elements of the spectral matrix to zero as presented in (25). However, it is crucial to include the affects of the off-diagonal terms into the stiffness matrix in order to generate an accurate model of the system. For this reason, while forming a total spectrum matrix, $\bar{\lambda}$ which is combining the first and second subsystems as one system, the eigenvalues of the diagonal elements $\bar{\lambda}_1$, $\bar{\lambda}_2$ will be shifted as $\Delta\lambda_i$ due to the dynamic forces which are occurred between the subsystems. Hence, the modes of the first subsystem are written as

$$\lambda_{1i} = \lambda_{1i}^{(0)} + \Delta\lambda_i \quad (53)$$

where $\lambda_{1i}^{(0)}$ represents the vibration modes of the first subsystem as if it is separate system, without any connection to the second subsystem, $\Delta\lambda_i$ is the shift and λ_{1i} is the new vibration modes of the first subsystems after the shifting procedure. Similarly, the modes of the second subsystem are computed by using the below equation

$$\lambda_{2i} = \lambda_{2i}^{(0)} + \Delta\lambda_i \quad (54)$$

where $\lambda_{2i}^{(0)}$ is the vibration modes of the first subsystem as if it is separate system and λ_{2i} is the new vibration modes of the second subsystems after the shifting procedure. The shift, $\Delta\lambda_i$ is described by the following equation

$$\Delta\lambda_i = \sum_{i=1}^k f_{i1} \cong \sum_{i=1}^k \frac{\lambda_{i1}}{\lambda_i^{(1)} - \lambda_i^{(2)}} \quad (55)$$

In the above summation, $\Delta\lambda_i$ will be maximum for the smaller vibration modes (the dominant modes) of the subsystems that are near to the first vibration modes of the first and second subsystems $\lambda_1^{(1)}$, $\lambda_1^{(2)}$ respectively and it will be minimum for the bigger vibration modes (the less dominant modes) that are far to the values of $\lambda_1^{(1)}$, $\lambda_1^{(2)}$. Hence, as the mode number, i increases the effect of the big modes decrease as follows

$$\frac{1}{(k^2 - 1)\omega_c^2} \cong \frac{1}{k^2} \quad (56)$$

As it is formulated in (52), the modes which have bigger mode numbers will not affect the system significantly therefore, those modes can be neglected and the system can be described by using the reduced number of modes. In practice, it is observed that this method is relevant and applicable to the gearing mechanisms of the machines due to the fact that even after the third natural frequency the effects of the vibration modes are very small therefore they can be ignored. Generally, the first three modes; $\lambda_1 \approx \lambda_c$, $\lambda_2 \approx 4\lambda_c$ and $\lambda_3 \approx 9\lambda_c$ are often used to describe the vibration systems in practice.

$$\lambda_k \gg \lambda_c \quad (57)$$

According to the above strong inequality, the vibration modes, λ_k which are very bigger than the characteristic vibration mode of the system, λ_c can be ignored effectively. This is an empirical method and it can be use to reduce the vibration modes of the large dynamic systems that consist of many branches. It has been observed practically that it is not necessary to consider all the vibration modes in order to model the system accurately. However, it is vital to retain the most dominant vibration modes of the subsystems which are active in the mid-frequency range.

7. Conclusions

In this paper, the reduced free vibration modes of the large branched gearing mechanism are analysed. The gearing mechanisms consist of two or more subsystems and each subsystem has many gears which are connected to each other. In real life, these gears work together in order to produce acceptable output results for a given task. The stiffness matrix of the system is very large due to the fact that the system is also very large. Hence, it is rather difficult and complicated to compute the vibration modes (the eigenvalues) of the system from this large matrix. For this reason, the system is divided into the subsystems. However, the stiffness matrix of each subsystem is still dense and the computation of the free vibration modes from this matrix is complicated and time consuming. In order to overcome these difficulties the dqds algorithm which is known good in finding the eigenvalues of large matrices is used to compute the free vibration modes of the subsystems. The method is applied to two branches gearing mechanism and it is generalized for the large systems which have more than two branches in this paper.

In order to analyse the free vibration modes of the large branched system each branch is considered as a subsystem and the vibration modes of each subsystem are computed separately. The dqds algorithm is used to analyse the free vibration modes of each subsystem. In the model not all the vibration modes of the subsystems are considered. The most dominant vibration modes which are corresponding to the mid-frequency range (usually they are the first three modes) are considered in the case of each subsystem. Generally, the big modes are not included into the dynamic models owing to the fact that they are the less dominant vibration modes therefore, those vibration modes are ignored. The most systems work in the mid-frequency range therefore the vibration modes which are corresponding to the high-frequency range will not be activated the most of the times. For this reason, if a system is not working in the high-frequency range then, the big modes will not effect the plant output response.

Although, the real system has a large number of vibration modes, the dynamic characteristics of the plant are mainly governed by lower frequency characteristics, *i. e.*, the lower order vibration modes. In this work, a new mode reduction method is proposed in order to simplify the dynamic model of the system. This is an empirical method and the application of the method to the large branched systems is rather easy. The method reduces the computation time and efforts great deal due to the fact that it retains only the most dominant modes of the system in the model which are active in the mid-frequency range. Hence, the less dominant vibration modes of the system which are coressponding to under or over range of the working frequencies can be ignored effectively by using this empirical mode reduction method and the model of the systems can be simplify a great deal.

The application of this new empirical method to a real large vibration system will be the subject of a formal analysis in future work on this topic. The new mode reduction method is also suitable to analyse the forced vibration modes of the large branched systems. In particular, the dynamic model of a real large branched system will be built by using the subsystems method. Then, the modes of the subsystems will be analysed by using the dqds algorithm and finally the

modes of each subsystems will be reduce by using the new mode reduction method. Later on, the results of the model and the real system will be compared in order see how effective, accurate and practical this method in real life applications.

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