

Performance Analysis of a Batch Arrival Queueing System Under Fuzzy Environment

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Abstract: In this paper, we investigate the steady-state performance analysis of a batch arrival queueing system $(M^X | M | 1)$ under fuzzy environment. In practice, operational parameters such as arrival and service rates are rarely known with precision. Randomness-Fuzziness consistency principle is used to define a normal law of fuzziness established by two distinct laws of randomness for the system's uncertain parameters. Two different models are constructed and analysed; the $(FM^X | M | 1)$ model, which feature fuzzy batch arrivals paired with service as a crisp value and $(M^X | FM | 1)$ model, where batches arrive at a crisp value but undergo service governed by a Triangular Fuzzy Number. This approach yields the membership functions for essential steady state system performance measures. Numerical example is solved to demonstrate the practical application and validity of the theoretical models.

Keywords: Batch arrival, $(M^X | M | 1)$ model, Triangular Fuzzy number, Randomness fuzziness consistency principle.

1. Introduction

In real-world manufacturing systems, telecommunication networks, computer processing architectures, and service industries, arrival patterns rarely manifest as single, perfectly spaced events. Instead, demands frequently occur in groups or bursts, a phenomenon mathematically modelled using batch arrival queues (M^X) . Traditional queueing frameworks rely heavily on classical probability theory, presupposing that system operational parameters such as arrival rates (λ) and service rates (μ) can be measured with precise numerical certainty. However, in practical engineering and economic landscapes, collecting exact probabilistic data is often restricted by incomplete information, fluctuating environment dynamics, or subjective human estimates. Relying solely on crisp values in these contexts can lead to misleading system performance evaluations and sub-optimal design decisions. To bridge this gap, fuzzy queueing models have emerged as an invaluable paradigm, mapping parameter ambiguity into mathematical possibilities. Analysis of the bulk queues was done by Dr. Yahya Mohamed, N. Karthikeyan in (2018) [17]. For a limited space correlated queueing problem with departures in batches of variable size discussed by Sharda in 1981 [16]. Erlang, A.K started to work on applying the theory of probabilities to problems of telephone traffic in (1909) [8] and in (1917) [9]. The steady-state system performance measures (such as expected queue length, waiting times, and server utilization) are evaluated. The study of bulk/batch arrival queueing systems has a rich history originating from the fundamental works of Chaudhry and Templeton in (1983) [5], who demonstrated that modeling arrivals in batches (M^X) provides a more realistic representation of packet transmissions and inventory logistics than simple Poisson arrivals. Donald Gross; Carl M. Harris in (1998) [11] also focused on the fundamentals of Queueing theory. Al-khedhairi and Tadj in (2007) [1] investigated the queueing process of a bulk service queueing system under Bernoulli schedule. The queueing process is studied both in discrete time and in continuous time. By utilizing Extension Principle of Zadeh in (1975) [20] and parametric programming techniques, the inherent uncertainty of the system parameters is thoroughly analyzed. Basis for a theory of possibility was analysed by Zadeh in (1978)

[18]. A core theoretical contribution of this work lies in the application of the Randomness-Fuzziness Consistency Principle. This principle allows us to systematically define a normal law of fuzziness established by two distinct underlying laws of randomness within the system's uncertain parameters. The introduction of fuzzy set theory by Zadeh in (1965) [19] fundamentally revolutionized how researchers approach non-probabilistic uncertainty. Prade (1980) [15] first introduced the concept of fuzzy queues, but early methods suffered from extreme mathematical complexity when attempting to compute the algebraic combinations of fuzzy numbers. Markovian bulk queues were developed by Mueen,Z .in (2024) [14] and the parametric approach to fuzzy queues with batch arrival and Erlang service was discussed by Chen, P. in (2015) [6]. A major breakthrough occurred when Kao, C., & Li, S. P.in (2002) [12] pioneered the use of Zadeh's Extension Principle paired with parametric programming nonlinear optimization techniques. This approach allowed researchers to decompose fuzzy queueing parameters into a family of crisp intervals via α -cuts, converting a complicated fuzzy problem into a solvable set of classical mathematical programs. Buckley, J.J in (1990) [4] evaluated fuzzy equations by using α -cuts.Following this methodology, numerous extensions have been developed. For instance, fuzzy batch arrival queues with generalized service patterns were explored by researchers like Ke,J.C., et al. (2007) [13].Furthermore, Goswami and Chaudhry in (2022)[10] provided explicit distribution results for batch arrivals transitioning into phase-type services($M^X|PH|1$), proving that phase segmentation remains the gold standard for representing deep service-layer mechanics.Despite these developments, a critical challenge persists in fuzzy queueing theory: reconciling the simultaneous coexistence of probabilistic randomness (stochastic customer behaviours) and epistemic fuzziness (imprecise data measurement). The Randomness-Fuzziness Consistency Principle provides the necessary mathematical bridge to handle this hybrid uncertainty. It was effectively done by H.K Baruah in 2010[2] and in 2015[3] and the parametric approach of Fuzzy queues by Randomness fuzziness consistency principle handled by Dhruba Das, H.K.Baruah in (2015) [7]. It establishes a coherent law of fuzziness derived systematically from underlying laws of randomness. While general service structures have been investigated under basic fuzzy assumptions, a dedicated exploration into a three-phase Erlangian distribution utilizing the Randomness-Fuzziness Consistency Principle remains present in literature. This study fills this distinct gap by isolating and analysing the separate impacts of parameter ambiguity through the ($FM^X|M|1$) and ($M^X|FM|1$) models. This dual-model perspective ensures that system designers can confidently evaluate steady-state metrics regardless of whether the primary source of fuzziness stems from the bulk arrival stream or from the multi-stage service engine.

2. Preliminaries

2.1 Queueing system

Queueing system can be described as customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served.The

Characteristics of queueing system are

1. Arrival (or inter-arrival) pattern of customers
2. Service pattern of customers
3. Queue discipline
4. Behaviour of customers
5. System capacity

2.2 Arrival Process

The arrival process defines how customers enter and engage with a queueing system. This pattern is typically quantified by the mean arrival rate, which represents the average volume of incoming units whether humans, machinery, or digital data per specific time interval. Alternatively, this process can be characterized by the mean inter-arrival time, which measures the chronological duration between consecutive arrivals. When the statistical

properties of this influx remain constant over time, the arrival pattern is classified as stationary; conversely, if the pattern fluctuates over time, it is designated as non-stationary.

2.3 Service process

The operational capacity of a server is defined by its service pattern, which can be quantified in two ways. It can be expressed as the service rate, representing the average quantity of customers processed within a specific timeframe. Alternatively, it can be measured by the inter-service time, which signifies the actual duration necessary to fulfill the service requirements of an individual customer.

2.4 Batch Arrival Queue

A batch (or bulk) arrival queue is a specialized queueing model designed for systems where entities enter collectively in groups rather than one by one. In standard Kendall's notation, these systems are identified by attaching a superscript X to the arrival distribution. For example, $M^X | M | 1$ or $M^X | G | 1$. Here, the batch size is treated as a random variable, denoted by X , which follows a discrete probability distribution:

$$P(X = k) = c_k, \quad k = 1, 2, 3, \dots \quad \text{and} \quad \sum_{k=1}^{\infty} c_k = 1$$

where c_k represents the probability that exactly k customers arrive together in a single batch.

2.5 $M^X | M | 1$ Queue

The $M^X | M | 1$ queueing model represents a single-server system in which M^X (Batch Arrivals): Customers arrive in groups (or batches) according to a Poisson process. The size of each arriving batch is a random variable, X and the server processes each individual customer through exponential distribution.

There is only one server facility, and it can only process one customer at a time.

In this model,

$$\text{Average number of customers } L = \frac{[\lambda E(X) + E(X^2)]}{2[\mu - \lambda E(X)]}$$

$$\text{Average waiting time } W = \frac{[E(X) + E(X^2)]}{2E(X)[\mu - \lambda E(X)]}$$

2.6 Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}$ is completely specified by a triplet of real numbers $\tilde{A} = (a, b, c)$ where $a \leq b \leq c$. The parameter a represents the lower limit, b represents the modal value (or peak), and c represents the upper limit of the fuzzy quantity with membership function defined as

$$\theta_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}; & a \leq x \leq b \\ \frac{c-x}{c-b}; & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

2.7 The α -Cut Representation (Interval Analysis)

A Triangular fuzzy number is often represented analytically via its α -cuts (where $\alpha \in [0, 1]$).

By setting the left and right branches of the membership function equal to α and solving for x the α cut of $\tilde{A}$ yields a closed crisp interval: $A_\alpha = [x_\alpha^L, x_\alpha^U] = [a + \alpha(b - a), c - \alpha(c - b)]$

2.8 Randomness-Fuzziness Consistency Principle

Let a variable Y be bounded within a fuzzy interval around a core value b , written as the normal fuzzy number $[a, b, c]$ and the corresponding membership function $\phi_Y(x)$,

$$\phi_Y(x) = \begin{cases} L(x), & \text{if } a \leq x \leq b \\ R(x), & \text{if } b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

The left reference function $L(x)$ is the distribution function in $[a, b]$ and the right reference function $R(x)$ is the complementary distribution function in the interval $[b, c]$ with

$L(a)=R(c)=0$, $L(b)=R(b)=1$ and the constant reference function equal to 0.

3 Model Description

3.1 The Model ($FM^X | M | 1$)

Batch arrival single server and general service queueing model is considered. The arrivals follow Poisson distribution as a fuzzy parameter ($\tilde{\lambda}_R$) and the service (μ) follows exponential distribution.

Parametrization of Fuzzy arrival rate via α -cuts

The fuzzy arrival rate be represented by $\tilde{\lambda}_R$, characterise by the set pairs

$$\tilde{\lambda}_R = \{M, \varphi_{\tilde{\lambda}_R}(M) | M \in R^+\}$$

Invoking Zadeh's extension principle, the membership function corresponding to the system performance measure, denoted as $\Omega_{g(\tilde{\lambda}_R, \mu)}(\varphi)$ and is formulated as

$$\Omega_{g(\tilde{\lambda}_R, \mu)}(\varphi) = \sup_{M \in R^+} \{\varphi_{\tilde{\lambda}_R}(M) | \varphi = g(M, \mu)\}$$

For a specific possibility level α , the structural boundaries of the fuzzy arrival rate $\tilde{\lambda}_R$ are captured by its α -cuts, which forms the following closed interval

$$\tilde{\lambda}_R = [M_\alpha^L, M_\alpha^U]$$

$$\tilde{\lambda}_R = \left[\min_{M \in R^+} \{M | \varphi_{\tilde{\lambda}_R}(M) \geq \alpha\}, \max_{M \in R^+} \{M | \varphi_{\tilde{\lambda}_R}(M) \geq \alpha\} \right]$$

Model Reduction and Interval Optimization

Through this approach, a fuzzy queueing framework can be systematically decomposed into a parametric family of classical $M^X | M | 1$ queues. The lower and upper limits of these parametric intervals vary as functions of α , expressed directly through the inverse membership functions such that

$$M_{\tilde{\lambda}_R}^L(\alpha) = \min \varphi_{\tilde{\lambda}_R}^{-1}(\alpha) \text{ and } M_{\tilde{\lambda}_R}^U(\alpha) = \max \varphi_{\tilde{\lambda}_R}^{-1}(\alpha)$$

Consequently, to evaluate the crisp boundaries of the performance objective function

$g(\tilde{\lambda}_R, \mu)$ at a given possibility threshold α , we establish the following pair of constrained optimization problems:

$$M_g^L(\alpha) = \min g(M, \mu) \text{ subject to } M_{\tilde{\lambda}_R}^L(\alpha) \leq M \leq M_{\tilde{\lambda}_R}^U(\alpha)$$

$$M_g^U(\alpha) = \max g(M, \mu) \text{ subject to } M_{\tilde{\lambda}_R}^L(\alpha) \leq M \leq M_{\tilde{\lambda}_R}^U(\alpha)$$

Derivation of Reference and Membership Functions

Assuming that the boundary functions $M_g^L(\alpha)$ and $M_g^U(\alpha)$ possess well defined inverses relative to α , we define the left and right shape function (reference functions) of the performance membership function $\Omega_{g(\tilde{\lambda}_R, \mu)}(\varphi)$ as

$$L(\varphi) = M_g^{L^{-1}}(\alpha) \text{ and } R(\varphi) = M_g^{U^{-1}}(\alpha)$$

By applying the randomness fuzziness consistency principle, the complete membership function for the system performance measure is synthesized as follows:

$$\Omega_{g(\tilde{\lambda}_R, \mu)}(\varphi) = \begin{cases} L(\varphi); & \varphi_1 \leq \varphi \leq \varphi_2 \\ R(\varphi); & \varphi_2 \leq \varphi \leq \varphi_3 \\ 0, & \text{otherwise} \end{cases}$$

Here the boundary conditions satisfy $L(\varphi_1) = R(\varphi_3) = 0$ and $L(\varphi_2) = R(\varphi_2) = 1$

Membership Functions for System Performance Measures

The membership functions characterizing the fuzzy expected queue length ($\tilde{L}_R$) and the fuzzy expected waiting time in the queue ($\tilde{W}_R$) are formulated via the supremum function as follow:

$$(i) \varphi_{\tilde{L}_R}(\Omega) = \sup_{\substack{(\tilde{\lambda}_R, \mu) \in \mathbb{R}^+ \text{ and} \\ \frac{\tilde{\lambda}_R E(X)}{\mu} < 1}} \left\{ \varphi_{\tilde{\lambda}_R}(M) \middle| \Omega = \left[\frac{\tilde{\lambda}_R [E(X) + E(X^2)]}{2[\mu - \tilde{\lambda}_R E(X)]} \right] \right\}$$

$$(ii) \varphi_{\tilde{W}_R}(\Omega) = \sup_{\substack{(\tilde{\lambda}_R, \mu) \in \mathbb{R}^+ \text{ and} \\ \frac{\tilde{\lambda}_R E(X)}{\mu} < 1}} \left\{ \varphi_{\tilde{\lambda}_R}(M) \middle| \frac{[E(X) + E(X^2)]}{2E(X)[\mu - \tilde{\lambda}_R E(X)]} \right\}$$

Where $\varphi_{\tilde{\lambda}_R}(M)$ denotes the membership function of the fuzzy arrival rate $\tilde{\lambda}_R$.

3.2 Numerical Example

Consider $\tilde{\lambda}_R = (2, 3, 4)$, $\mu = 8$ per hour, X takes the values 1, 2 and 3 with corresponding probabilities 0.5, 0.3 and 0.2.

Solution: $\tilde{\lambda}_R = (2, 3, 4)$, $\mu = 8$ per hour, $E(X) = 1.7$, $E(X^2) = 3.5$

By using α -cut interval,

The confidence interval of the arrival rate is

$$\tilde{\lambda}_R(\alpha) = [\tilde{\lambda}_R^L(\alpha), \tilde{\lambda}_R^U(\alpha)]$$

$$\tilde{\lambda}_R^L(\alpha) = 2 + \alpha, \quad \tilde{\lambda}_R^U(\alpha) = 4 - \alpha \quad \text{and} \quad \tilde{\lambda}_R(\alpha) = (2 + \alpha, 4 - \alpha)$$

Now the membership function of $\tilde{L}_R$ by the parametric approach is

$$\Omega_{\tilde{L}_R}^L = \min \left[\frac{\tilde{\lambda}_R [E(X) + E(X^2)]}{2[\mu - \tilde{\lambda}_R E(X)]} \right] \text{ subject to } 2 + \alpha \leq M \leq 4 - \alpha \text{ -----(1)}$$

$$\Omega_{\tilde{L}_R}^U = \max \left[\frac{\tilde{\lambda}_R [E(X) + E(X^2)]}{2[\mu - \tilde{\lambda}_R E(X)]} \right] \text{ subject to } 2 + \alpha \leq M \leq 4 - \alpha \text{ -----(2)}$$

The solution of (1) when $\tilde{\lambda}_R = 2 + \alpha$ is minimum,

$$\Omega_{\tilde{L}_R(\alpha)}^L = \left[\frac{(2+\alpha)[1.7+3.5]}{2[8-(2+\alpha)1.7]} \right], \quad \Omega_{\tilde{L}_R(\alpha)}^L = \left[\frac{2.6(2+\alpha)}{[8-(2+\alpha)1.7]} \right]$$

The solution of (2) when $\tilde{\lambda}_R = 4 - \alpha$ is maximum, $\Omega_{\tilde{L}_R(\alpha)}^U = \left[\frac{2.6(4-\alpha)}{[8-(4-\alpha)1.7]} \right]$

Since the inverse of $\Omega_{\tilde{L}_R(\alpha)}^L$ and $\Omega_{\tilde{L}_R(\alpha)}^U$ exists, they yield the left and right reference functions of membership function $\chi_{\tilde{L}_R}(\varphi)$ as

$$\chi_{\tilde{L}_R}(\varphi) = \begin{cases} L(\varphi) = \frac{4.6\varphi - 5.2}{2.6\varphi + 1.7}, & 1.1304 \leq \varphi \leq 2.6897 \\ R(\varphi) = \frac{10.4 - 1.2\varphi}{2.6\varphi + 1.7}, & 2.6897 \leq \varphi \leq 8.6667 \\ 0, & \text{otherwise} \end{cases}$$

By Randomness-Fuzziness Consistency Principle,

$L(\varphi), 1.1304 \leq \varphi \leq 2.6897$ and $R(\varphi), 2.6897 \leq \varphi \leq 8.6667$

The calculated values are given by the table

α	$\tilde{L}_R^L$	$\tilde{L}_R^U$	$\tilde{W}_R^L$	$\tilde{W}_R^U$
0.0	1.1304	8.6667	0.3325	
0.1	1.2325	7.8000	0.3452	1.1164
		7.0649	0.3590	0.9931
		6.4327	0.3739	
		5.8830	0.3902	
		5.4010	0.4078	
		4.9752	0.4272	
		4.5966	0.4485	
		4.2578	0.4720	0.5974
		3.9524	0.4982	0.5602
		2.6897	0.5274	0.5274

Table (3.2.1)

Hence the membership curve of average length of queue is

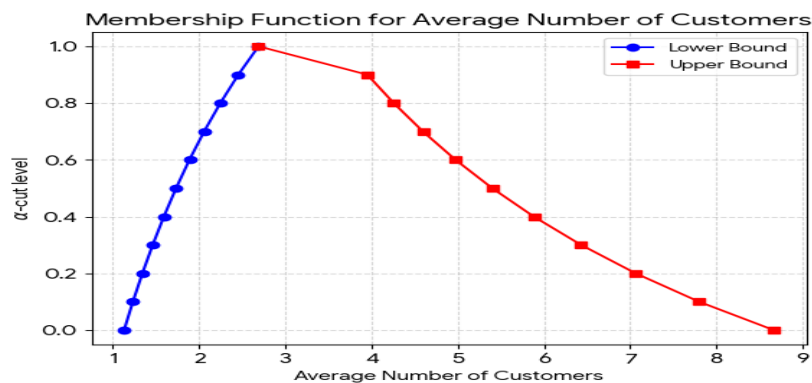


Figure: (3.2.2): The membership curve of average length

The membership function curve shows an asymmetrical triangular-like distribution, indicating a right-skewed uncertainty profile. At the lowest level of membership ($\alpha = 0.0$), the average number of customers fluctuates across a broad spectrum, ranging from a minimum of 1.1304 to a maximum of 8.6667. As the confidence level increases ($\alpha \rightarrow 1$), the upper and lower boundaries steadily converge toward the crisp value of 2.6897 customers. This indicates that under stable, highly predictable operational conditions, the system maintains a manageable, moderate number of accumulated individuals.

Now the membership function of $\tilde{W}_R$ by the parametric approach are

$$\Omega_{\tilde{W}_R}^L = \min \left[\frac{[E(X) + E(X^2)]}{2E(X)[\mu - \tilde{\lambda}_R E(X)]} \right] \text{ subject to } 2 + \alpha \leq M \leq 4 - \alpha \text{----- (3)}$$

$$\Omega_{\tilde{W}_R}^U = \max \left[\frac{[E(X) + E(X^2)]}{2E(X)[\mu - \tilde{\lambda}_R E(X)]} \right] \text{ subject to } 2 + \alpha \leq M \leq 4 - \alpha \text{----- (4)}$$

The solution of (3) when $\tilde{\lambda}_R = 2 + \alpha$ is minimum,

$$\Omega_{\tilde{W}_R(\alpha)}^L = \left\lfloor \frac{1.5294}{[8-(2+\alpha)(1.7)]} \right\rfloor$$

The solution of (4) when $\tilde{\lambda}_R = 4 - \alpha$ is maximum,

$$\Omega_{\tilde{W}_R(\alpha)}^U = \left\lfloor \frac{1.5294}{[8-(4-\alpha)(1.7)]} \right\rfloor$$

Since the inverse of $\Omega_{\tilde{W}_R(\alpha)}^L$ and $\Omega_{\tilde{W}_R(\alpha)}^U$ exists, they yield the left and right reference functions of membership function $\chi_{\tilde{W}_R}(\varphi)$ as

$$\chi_{\tilde{W}_R}(\varphi) = \begin{cases} L(\varphi) = \frac{4.6\varphi - 1.5294}{1.7\varphi}, & 0.3324 \leq \varphi \leq 0.5274 \\ R(\varphi) = \frac{1.5294 - 1.2\varphi}{1.7\varphi}, & 0.5274 \leq \varphi \leq 1.2745 \\ 0, & \text{otherwise} \end{cases}$$

$L(\varphi)$ and $R(\varphi)$ are the left and right reference functions and they are distribution and complementary distribution by the randomness-fuzziness consistency principle. Now the membership curve for the average waiting time in the queue is

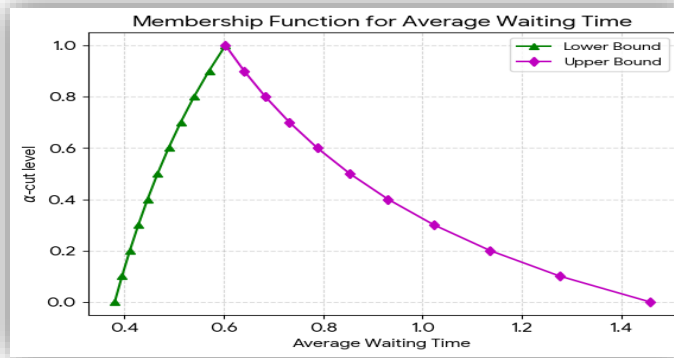


Figure: 3.2.3: The membership curve for the average waiting time

The membership function curve for the average waiting time is bounded between 0.3804 and 1.4583 time units. The narrow width (a total spread of 1.0779) suggests that even when input parameters fluctuate drastically, the actual duration a customer spends waiting remains stable and predictable. At $\alpha = 1.0$, yielding a crisp average waiting time of 0.6034 time units. This indicates high resilience in scheduling, ensuring that service delays do not scale out of control even under maximum system stress.

4 Model Description

4.1 The Model (M^X FM 1)

In this model arrivals follow Poisson distribution with a parameter (λ) and the service follows exponential distribution with fuzzy parameter $\tilde{\mu}_R$.

Parametrization of Fuzzy service rate via α -cuts

The fuzzy service rate be represented by $\tilde{\mu}_R$, characterise by the set pairs

$$\tilde{\mu}_R = \{M, \varphi_{\tilde{\mu}_R}(M) | M \in R^+\}$$

By Zadeh's extension principle, the membership function corresponding to the system performance measure, denoted as $\Omega_{g(\lambda, \tilde{\mu}_R)}(\varphi)$ and is formulated as

$$\Omega_{g(\lambda, \tilde{\mu}_R)}(\varphi) = \sup_{M \in R^+} \{\chi_{\tilde{\mu}_R}(M) | \varphi = g(\lambda, M)\}$$

For a specific possibility level α , the structural boundaries of the fuzzy service rate $\tilde{\mu}_R$ are captured by its α -cuts, which forms the following closed interval

$$\tilde{\mu}_R = [M_\alpha^L, M_\alpha^U]$$

$$\tilde{\mu}_R = \left[\min_{M \in \mathbb{R}^+} \{M | \varphi_{\tilde{\mu}_R}(M) \geq \alpha\}, \max_{M \in \mathbb{R}^+} \{M | \varphi_{\tilde{\mu}_R}(M) \geq \alpha\} \right]$$

Model Reduction and Interval Optimization

Through this approach, a fuzzy queueing framework can be systematically decomposed into a parametric family of classical $M^X | FM | 1$ queues. The lower and upper limits of these parametric intervals vary as functions of α , expressed directly through the inverse membership functions such that

$$M_{\tilde{\mu}_R}^L(\alpha) = \min \varphi_{\tilde{\mu}_R}^{-1}(\alpha) \quad \text{and} \quad M_{\tilde{\mu}_R}^U(\alpha) = \max \varphi_{\tilde{\mu}_R}^{-1}(\alpha)$$

Consequently, to evaluate the crisp boundaries of the performance objective function

$g(\lambda, \tilde{\mu}_R)$ at a given possibility threshold α , we establish the following pair of constrained optimization problems:

$$M_g^L(\alpha) = \min g(\lambda, M) \text{ subject to } M_{\tilde{\mu}_R}^L(\alpha) \leq M \leq M_{\tilde{\mu}_R}^U(\alpha)$$

$$M_g^U(\alpha) = \max g(\lambda, M) \text{ subject to } M_{\tilde{\mu}_R}^L(\alpha) \leq M \leq M_{\tilde{\mu}_R}^U(\alpha)$$

Derivation of Reference and Membership Functions

Assuming that the boundary functions $M_g^L(\alpha)$ and $M_g^U(\alpha)$ possess well defined inverses relative to α , we define the left and right shape profiles (reference functions) of the performance membership function $\Omega_{g(\lambda, \tilde{\mu}_R)}(\varphi)$ as

$$L(\varphi) = M_g^{L^{-1}}(\alpha) \text{ and } R(\varphi) = M_g^{U^{-1}}(\alpha)$$

By applying the randomness fuzziness consistency principle, the complete membership function for the system performance measure is synthesized as follows:

$$\Omega_{g(\lambda, \tilde{\mu}_R)}(\varphi) = \begin{cases} L(\varphi); & \varphi_1 \leq \varphi \leq \varphi_2 \\ R(\varphi); & \varphi_2 \leq \varphi \leq \varphi_3 \\ 0, & \text{otherwise} \end{cases}$$

Here the boundary conditions satisfy $L(\varphi_1) = R(\varphi_3) = 0$ and $L(\varphi_2) = R(\varphi_2) = 1$

Membership Functions for System Performance Measures

The membership functions characterizing the fuzzy expected queue length $\tilde{L}_R$ and the fuzzy expected waiting time in the queue $\tilde{W}_R$ are formulated via the supremum profile as follow:

$$\varphi_{\tilde{L}_R}(\Omega) = \sup_{\substack{(\lambda, \tilde{\mu}_R) \in \mathbb{R}^+ \text{ and} \\ \frac{\lambda E(X)}{\tilde{\mu}_R} < 1}} \left\{ \varphi_{\tilde{\mu}_R}(M) \left| \left[\frac{\lambda[E(X) + E(X^2)]}{2[\tilde{\mu}_R - \lambda E(X)]} \right] \right. \right\}$$

$$\varphi_{\tilde{W}_R}(\Omega) = \sup_{\substack{(\lambda, \tilde{\mu}_R) \in \mathbb{R}^+ \text{ and} \\ \frac{\lambda E(X)}{\tilde{\mu}_R} < 1}} \left\{ \varphi_{\tilde{\mu}_R}(M) \left| \left[\frac{\lambda[E(X) + E(X^2)]}{2E(X)[\tilde{\mu}_R - \lambda E(X)]} \right] \right. \right\}$$

Where $\varphi_{\tilde{\mu}_R}(M)$ denotes the membership function of the fuzzy service rate $\tilde{\mu}_R$.

4.2 Numerical Example

Consider $\lambda = 3$, $\tilde{\mu}_R = (7, 8, 9)$ and the random variable X take the same values discussed in section 3.2.

Solution:

$\lambda = 3$ batches per hour

The triangular fuzzy number (d, e, f) for the service rate is $\tilde{\mu}_R = (7, 8, 9)$

$E(X) = 1.7$ and $E(X^2) = 3.5$

By using α -cut interval, the confidence interval of the arrival rate is

$$\tilde{\mu}_R(\alpha) = [\tilde{\mu}_R^L(\alpha), \tilde{\mu}_R^U(\alpha)]$$

$$\tilde{\mu}_R^L(\alpha) = 7 + \alpha, \quad \tilde{\mu}_R^U(\alpha) = 9 - \alpha$$

$$\tilde{\mu}_R(\alpha) = (7 + \alpha, 9 - \alpha)$$

Now the membership function of $\tilde{L}_{q_R}$ by the parametric approach is

$$\Omega_{L_R}^L = \min \left[\frac{\lambda[E(X) + E(X^2)]}{2[\tilde{\mu}_R - \lambda E(X)]} \right] \text{ subject to } 7 + \alpha \leq M \leq 9 - \alpha$$

----- (5)

$$\Omega_{L_R}^U = \max \left[\frac{\lambda[E(X) + E(X^2)]}{2[\tilde{\mu}_R - \lambda E(X)]} \right] \text{ subject to } 7 + \alpha \leq M \leq 9 - \alpha$$

----- (6)

The solution of (5) when $\tilde{\mu}_R = 8 + \alpha$ is minimum, $\Omega_{L_R}^L(\alpha) = \frac{7.8}{[(7+\alpha)-5.1]}$

The solution of (6) when $\tilde{\mu}_R = 9 - \alpha$ is maximum, $\Omega_{L_R}^U(\alpha) = \frac{7.8}{[(9-\alpha)-5.1]}$

Since the inverse of $\Omega_{L_R}^L(\alpha)$ and $\Omega_{L_R}^U(\alpha)$ exists, they yield the left and right reference functions of membership function $\chi_{L_R}(\varphi)$ as

$$\chi_{L_R}(\varphi) = \begin{cases} L(\varphi) = 3.9 - \frac{7.8}{\varphi}, & 2 \leq \varphi \leq 2.6897 \\ R(\varphi) = \frac{7.8}{\varphi} - 1.9, & 2.6897 \leq \varphi \leq 4.1052 \\ 0, & \text{otherwise} \end{cases}$$

By Randomness-Fuzziness Consistency Principle,

$L(\varphi), 2 \leq \varphi \leq 2.6897$ and $R(\varphi), 2.6897 \leq \varphi \leq 4.1052$

The calculated values are given by the table

α	$\tilde{L}_{q_R}^L$	$\tilde{L}_{q_R}^U$	$\tilde{W}_{q_R}^L$	$\tilde{W}_{q_R}^U$
0.0	2.0000	4.1053	0.3922	
0.1	2.0526	3.9000	0.4025	0.7647
		3.7143	0.4134	0.7283
		3.5455	0.4248	
		3.3913	0.4370	
		3.2500	0.4498	
		3.1200	0.4635	
		3.0000	0.4779	0.5882

		2.8889	0.4934	0.5664
		2.7857	0.5098	0.5462
		2.6897	0.5274	0.5274

Table 4.2.1

Hence the membership curve of average number of customers is

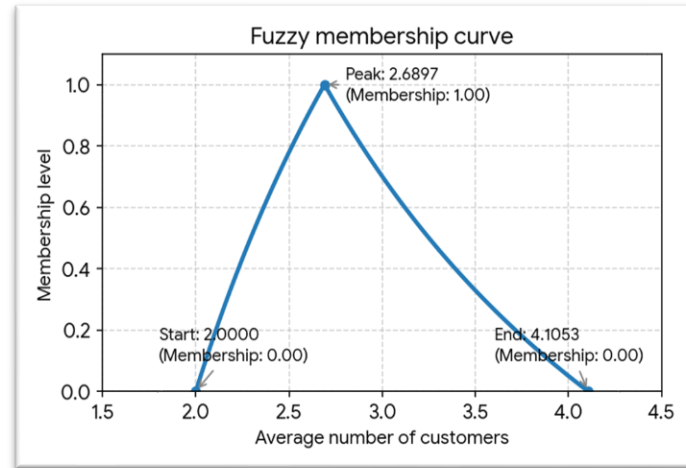


Figure: 4.2.2: The membership curve of average number of customers

Under steady-state conditions, the expected number of customers waiting in the queue is exactly 2.6897 (where the membership level is 1.00). Due to the system's parameters and uncertainty bounds, the average queue length will never drop below 2.0000 and will never exceed 4.1053 under any possible α -cut. The curve leans toward the lower bound. This indicates that while the queue length can quickly drop down toward 2.0000, it has a longer, more gradual "tail" extending toward 4.1053. This shows that under higher uncertainty levels, the queue length is prone to stretching out significantly before stabilizing.

Now the membership function of $\tilde{W}_R$ by the parametric approach are

$$\Omega_{\tilde{W}_R}^L = \min \left[\frac{[E(X) + E(X^2)]}{2E(X)[\tilde{\mu}_R - \lambda E(X)]} \right] \text{ subject to } 7 + \alpha \leq M \leq 9 - \alpha \quad (7)$$

$$\Omega_{\tilde{W}_R}^U = \max \left[\frac{[E(X) + E(X^2)]}{2E(X)[\tilde{\mu}_R - \lambda E(X)]} \right] \text{ subject to } 7 + \alpha \leq M \leq 9 - \alpha \quad (8)$$

The solution of (7) when $\tilde{\mu}_R = 7 + \alpha$ is minimum, $\Omega_{\tilde{W}_R(\alpha)}^L = \frac{1.5294}{[(7+\alpha)-5.1]}$

The solution of (8) when $\tilde{\mu}_R = 9 - \alpha$ is maximum, $\Omega_{\tilde{W}_R(\alpha)}^U = \frac{1.5294}{[(9-\alpha)-5.1]}$

Since the inverse of $\Omega_{\tilde{W}_R(\alpha)}^L$ and $\Omega_{\tilde{W}_R(\alpha)}^U$ exists, they yield the left and right reference functions of membership function $\chi_{\tilde{W}_R}(\varphi)$ as

$$\chi_{\tilde{W}_R}(\varphi) = \begin{cases} L(\varphi) = 3.9 - \frac{1.5294}{\varphi}, & 0.3922 \leq \varphi \leq 0.5274 \\ R(\varphi) = \frac{1.5294}{\varphi} - 1.9, & 0.5274 \leq \varphi \leq 0.8049 \\ 0, & \text{otherwise} \end{cases}$$

By Randomness-Fuzziness Consistency Principle,

$$L(\varphi), 0.3922 \leq \varphi \leq 0.5274 \text{ and } R(\varphi), 0.5274 \leq \varphi \leq 0.8049$$

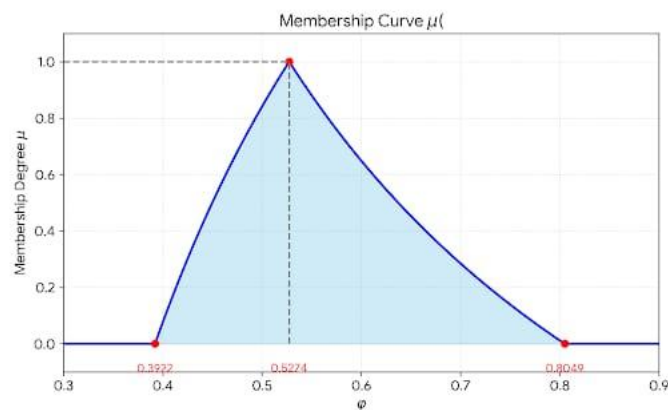


Figure 4.2.3: The membership curve of average waiting time

Hence the membership curve of average waiting time is

The most realistic, expected average time a customer can expect to wait in the queue under steady-state conditions is exactly 0.5274 time units (where the membership level is 1.00). Factoring in all levels of parameter uncertainty, the absolute minimum average waiting time is bounded at 0.3922, while the absolute maximum worst-case average waiting time is capped at 0.8049. Similar to the queue length, the waiting time curve exhibits a non-linear, right-skewed profile. The steep climb from 0.3922 to the peak shows that a minimum waiting time threshold is quickly established, whereas the prolonged right tail indicates that system delays decay slowly, making the system highly sensitive to variations that push waiting times toward the upper limit of 0.8049.

5 Performance Analysis

Regime	Service rate	ρ max	L core	L spread	L rel.	W core	W spread	W rel.
Light	$\mu=12$	0.5667	1.1304	1.3953	1.234	0.2217	0.1163	0.525
Moderate*	$\mu=8$	0.8500	2.6897	7.5362	2.802	0.5274	0.9420	1.786
Heavy	$\mu=7$	0.9714	4.1053	50.5556	12.315	0.8050	7.2222	8.972

Table 5.1: Model I — fixed $\tilde{\lambda}_R = (2, 3, 4)$, crisp μ varied.

Regime	Arrival rate	ρ max	L core	L spread	L rel.	W core	W spread	W rel.
Light	$\lambda=1$	0.2429	0.4127	0.1344	0.326	0.2428	0.0791	0.326
Moderate*	$\lambda=3$	0.7286	2.6897	2.1053	0.783	0.5274	0.4128	0.783
Heavy	$\lambda=4$	0.9714	8.6667	47.2727	5.455	1.2745	6.9519	5.455

Table 5.2: Model II — fixed $\tilde{\mu}_R = (7, 8, 9)$, crisp λ varied

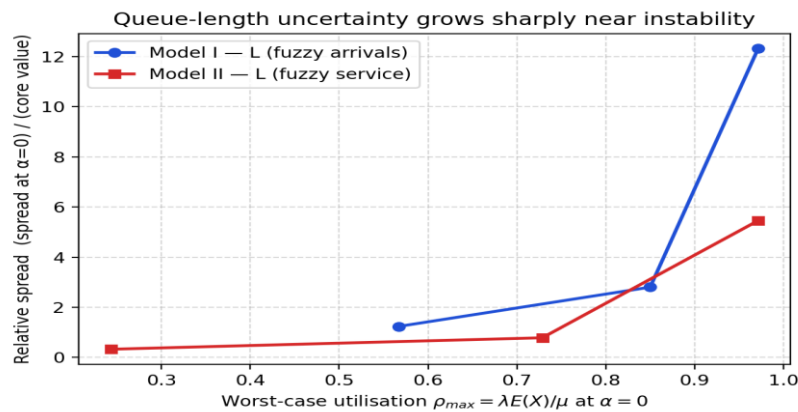


Fig 5.3: Analysis of Model I and Model II

From the above figure, relative spread grows sharply as ρ_{max} approaches 1, and does so for both L and W, not just the single pair reported in previous section. In Model I, relative spread rises from 1.23 (light load) to 2.80 (moderate) to 12.31 (heavy). In Model II it rises from 0.33 to 0.78 to 5.45. Both models converge to the identical crisp solution at $\alpha = 1$, confirming the internal consistency of the two constructions. Their behaviour diverges sharply, however, as α decrease toward 0. Under the original moderate-load parameters, the $\alpha = 0$ spread of the queue-length membership function is 7.5362 for Model I but only 2.1053 for Model II.

In Model I, the upper bound of the arrival rate ($\tilde{\lambda}_R = 4$) drives the worst-case utilisation to $\rho_{max} = 4 \times 1.7 / 8 = 0.85$, close to 1 that the queue-length formula's $(\mu - \lambda E(X))$ denominator becomes small and the performance measure is amplified sharply. In Model II, the corresponding worst case ($\tilde{\mu}_R = 7$) gives $\rho_{max} = 3 \times 1.7 / 7 \approx 0.729$, further from instability, so the same absolute uncertainty in the fuzzy parameter produces a more moderate values. The system performance is more sensitive to uncertainty on whichever side of the queue arrivals or service shows closer to the stability boundary at its extreme, and this holds generally across load regimes, not only under one convenient choice of numbers. For a system operating with a comfortable service margin, fuzzifying the arrival rate will tend to produce wider, more cautionary performance bounds than fuzzifying the service rate by a comparably-sized fuzzy number. This suggests that in applying this framework to a real system, the modelling effort which parameter to treat as fuzzy, and how tightly to bound it should be prioritised according to which parameter the system's stability margin is more sensitive to, with extra caution warranted precisely in the heavily-loaded regimes where that sensitivity is largest.

6 Conclusion

In this paper, a batch arrival queueing system with exponential service distribution has been successfully analysed under the framework of the Randomness-Fuzziness Consistency Principle. By systematically varying the membership levels, the epistemic vagueness of the fuzzy parameter was mapped into performance-measure membership functions for the expected number of customers in the system and expected time in the system, in both models. The exact convergence of the upper and lower boundary intervals onto a single crisp core value at $\alpha = 1$ confirms that the fuzzy models remain perfectly consistent with the underlying probabilistic model at the point of full confidence. The comparative analysis shows that the two models are not interchangeable in practice. This underlying finds that fuzzy parameter variations are non-linear and asymmetric, and depend on proximity to the system's stability boundary. While the membership curves of numerical problem in the model-I serves as an excellent foundational tool for assessing strict, step by step performance bounds under linear interval variations, the membership curves in the model-II offers a far more mature and realistic operational diagnostic. It clearly communicates that system failure or system degradation risks are non-linear, heavily skewed to the right and require asymmetrical buffers to maintain stability.

7 References

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