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An Energy Framework For Spherical Fuzzy Rough Matrices and Its Use in Multi-Criteria Decision-Making

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Abstract :- This study proposes the basic idea underlying rough set serves as a technique for man aging information in relational databases. It is a distinctive field of uncertainity mathematics, strongly associated with fuzzy set theory. The integration of set theory in rough form along with spherical fuzzy set leads to the development of spherical fuzzy rough matrices, a powerful tool for managing uncertainity and vagueness in complex data environments. While spherical fuzzy sets enhance the descriptive power of conventional fuzzy models by representing the degrees of belonging, non-belonging and indeterminacy within a spherical framework, rough sets provide boundary-based approximations to handle indiscernibility in data. The combination of these two frameworks allows spherical fuzzy rough matrices (SFRM) to represent uncertain, imprecise and incomplete information in a more robust manner. This study demonstrates that the spherical fuzzy rough matrix acts as a crucial tool in strategic selection processes. The determinant and adjoint of the proposed matrix are formulated, and a ranking function based on its energy is derived. The study introduces an MCDM-based framework for the evaluation and ranking of different alternatives. To validate the effectiveness of the approach, numerical examples are presented, showing the practical applicability of the SFRM and its energy in addressing MCDM problems.

Keywords: Rough matrix, Spherical fuzzy matrix, Spherical fuzzy rough matrix.

1. Introduction

In 1965, Zadeh pioneered the ideas of fuzzy sets, fuzzy membership functions and fuzzy logic [26] in 1965. In 1986, Atanassov [7] proposed intuitionistic fuzzy sets, broadening fuzzy set theory to incorporate both degree of belonging and degree of non-belonging for an improved uncertainity handling. Cuong [10,11] proposed Picture fuzzy sets, adding a neutrality dimension. Yager [24,25] proposed Pythagorean fuzzy sets, expanding the scope. Ashraf [1–6] proposed spherical fuzzy sets to tackle uncertainity, applying it to decision dilemmas. Matrix theory models relationships between objects and attributes, playing a crucial role in science, engineering and medicine. Thomason [21] pioneered fuzzy matrix theory in 1977. Fuzzy matrices have been extensively explored for their ability to model uncertainity and imprecision. Each element in a fuzzy matrix represents either a fuzzy number or a membership value, typically between 0 and 1, indicating degrees of truth membership or uncertainity. Such matrices are valuable tools for dealing with vague data. Pal et al. [16] explored and developed on this, introducing intuitionistic fuzzy matrices. Intuitionistic fuzzy matrices were developed to represent intuitionistic fuzzy relations finite universes, capturing the imprecise or uncertain relationships among their components. To enhance the representation of uncertainity beyond the conventional fuzzy framework, Dogra and Pal [12] proposed the notion of a picture fuzzy matrix, which extends the idea of fuzzy matrices to include additional degrees of truth, falsity and neutrality. To further generalize the existing fuzzy frameworks, the concept of the spherical fuzzy matrix [20] was introduced, which extends the traditional fuzzy matrix by incorporating the belongingness degree, falsity degree and ambiguity degrees under a spherical requirement. Pawlak [17,18] developed the rough set approach in 1982, deals with approximation based operations and set inclusions defined through relations. In 1985, rough sets were compared with fuzzy sets and later Dubois and Prade [13] in 1990 proposed two hybrid frameworks-rough fuzzy sets and fuzzy rough sets-for handling imprecise information. To manage partial information systems, Kryszkiewicz [15] proposed an extension of rough set theory, providing a foundation for handling uncertainity in data. Subsequently, Wu et al. [23] in 2003 proposed the generalized fuzzy rough set model, which further enhanced the capability of rough sets in dealing with vagueness and imprecision. Christi DiStefano et al. [9] proposed matrix energy in 2009 and formulated its mathematical expression, providing a generalization of graph energy. Subsequently, Bravo et al. [8] in 2017 conducted a comprehensive study titled "Energy of Matrices", in which they established several theorems on matrix energy and derived its boundary values. Vijayabalaji and Balaji [22] proposed rough matrix theory in 2013 and applied it to decision-making problems. This research focuses on determinant, adjoint and ranking functions of the spherical fuzzy rough matrices. A numerical example is provided to illustrate the MCDM procedure.

2. Preliminaries

Definition 2.1. [14,19] Let $\widehat{U_ɹ}$ denote the universal set and $\widehat{R_ɹ}$ be an equivalence relation on $\widehat{U_ɹ}$. The set of all equivalence classes generated by $R_ɹ$. is referred to as the approximation space, represented by $R_ɹ = \widehat{U_ɹ} / \widehat{R_ɹ}$. For a subset $X_ɹ \subseteq \widehat{U_ɹ}$, the lower and upper approximations of $X_ɹ$ within the space $R_ɹ$ are denoted by $\underline{R_ɹ}(X_ɹ)$ and $\overline{R_ɹ}(X_ɹ)$.

$$\underline{R_ɹ}(X_ɹ) = \left\{r \in \widehat{U_ɹ} : [r]_{\widehat{R_ɹ}} \subseteq X_ɹ\right\},$$

$$\overline{R_ɹ}(X_ɹ) = \{r \in \widehat{U_ɹ} : [r]_{\widehat{R_ɹ}} \cap X_ɹ = \varphi\},$$

Where $[r]_{\widehat{R_ɹ}}$ represents the equivalence class under the relation $\widehat{R_ɹ}$ that includes the element r. The ordered pair $R_ɹ(X) = (\underline{R_ɹ}(X_ɹ), \overline{R_ɹ}(X_ɹ)$ represents the rough set of $X_ɹ$ in $R_ɹ$.

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Definition 2.2. [2] A Spherical fuzzy set (SFS) on a universe $\widehat{U_ɹ}$ is defined as,

$$S = \{(\hat{x}, \widehat{\chi_S}(\hat{x}), \widehat{\tau_S}(\hat{x}), \widehat{\psi_S}(\hat{x})) \mid \hat{x} \in \widehat{U_ɹ}\}$$

For each element x in $\widehat{U_ɹ}$, $\widehat{\chi_S}(\hat{x}) \in [0,1]$, $\widehat{\tau_S}(\hat{x}) \in [0,1]$, $\widehat{\psi_S}(\hat{x}) \in [0,1]$ and $0 \leq \widehat{\chi_S}^2(\hat{x}) + \widehat{\tau_S}^2(\hat{x}) + \widehat{\psi_S}^2(\hat{x}) \leq 1$, $\forall\, \hat{x} \in \widehat{U_ɹ}$, where $\widehat{\chi_S}(\hat{x})$, $\widehat{\tau_S}(\hat{x})$, $\widehat{\psi_S}(\hat{x})$ are the degree of belongingness, ambiguity and falsity functions.

Definition 2.3. [14,22] Let $R_M = [\widehat{r_{ij}}]$ rough matrix of order $m \times n$ is defined as follows:

$$R_M = [\widehat{r_{ij}}] = \begin{bmatrix} \widehat{r_{11}} & \widehat{r_{12}} & \cdots & \widehat{r_{1n}} \\ \widehat{r_{21}} & \widehat{r_{22}} & \cdots & \widehat{r_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{r_{m1}} & \widehat{r_{m2}} & \cdots & \widehat{r_{mn}} \end{bmatrix}$$

where, each $\widehat{r_{ij}} \in {\mu_x}^R$. ${\mu_x}^R$ is rough membership function.

Definition 2.4. [20] Let $\hat{A} = (\widehat{\chi_{ij}^2}, \widehat{\tau_{ij}^2}, \widehat{\Psi_{ij}^2})$ be the non-negative real numbers of spherical fuzzy matrix, where $\widehat{\chi_{ij}} \in [0,1]$, $\widehat{\tau_{ij}} \in [0,1]$, $\widehat{\psi_{ij}} \in [0,1]$ that satisfies the condition

$0 \leq \widehat{\chi_{ij}}^2(\hat{x}) + \widehat{\tau_{ij}}^2(\hat{x}) + \widehat{\psi_{ij}}^2(\hat{x}) \leq 1$, $\forall\, i, j$.

Here $\widehat{\chi_{ij}} \in [0,1]$, $\widehat{\tau_{ij}} \in [0,1]$, $\widehat{\psi_{ij}} \in [0,1]$ be the degree of belongingness, ambiguity and falsity functions.

Definition 2.5. [8] Let $M_{n(C)}$ denote the space of $n \times n$ matrices with entries in C and P be a matrix in $M_{n(C)}$. We define the energy of A as

$$E(P) = \sum_{i=1}^{n} | \widehat{\lambda_\imath} - \widehat{\mu} |$$

3. Determinant-Based and Energy Measures for SFRM's

Definition 3.1. A Spherical fuzzy determinant (SFD) function $f_{\lambda}: \widehat{X_{\lambda}} \to \widehat{U_{\lambda}}$ is a function on the set $\widehat{X_{\lambda}}$ of $SFMs$ of order $n \times n$ over $\widehat{U_{\lambda}}$ such that if $\widehat{A_{\lambda}} \in \widehat{X_{\lambda}}$ then $f_{\lambda}(\widehat{A_{\lambda}})$ or $|\widehat{A_{\lambda}}|$ belongs to $\widehat{U_{\lambda}}$ and is given by

$$|\widehat{A_{\lambda}}| = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i_{\sigma(i)}}$$

Where $a_{i_{\sigma(i)}} = (a_{i_{\sigma(i)_{\widehat{\chi}}}}, a_{i_{\sigma(i)_{\widehat{\tau}}}}, a_{i_{\sigma(i)_{\widehat{\psi}}}})$ and S_n denotes the symmetric group of all permutations of the symbols $\{1,2,\ldots n\}$.

Definition 3.2. The spherical fuzzy matrix is represented by three matrices of truth $(\widehat{a_{\imath\jmath}} \in P(\widehat{\chi_{\imath\jmath}}))$, indeterminacy $(\widehat{b_{\imath\jmath}} \in P(\widehat{\tau_{\imath\jmath}}))$, and falsity $(\widehat{c_{\imath\jmath}} \in P(\widehat{\psi_{\imath\jmath}}))$ membership values. It is denoted as

$$P(S) = (P(\widehat{\chi_{\imath\jmath}}), P(\widehat{\tau_{\imath\jmath}}), P(\widehat{\psi_{\imath\jmath}}))$$

Then the spherical fuzzy matrix's energy is defined as

$E[P(S)] = (\mathrm{E}[P(\widehat{\chi_{\imath\jmath}})], \mathrm{E}[P(\widehat{\tau_{\imath\jmath}})], \mathrm{E}[P(\widehat{\psi_{\imath\jmath}})])$

Where $\widehat{\lambda_\imath}$, $\widehat{\varsigma_\imath}$,and $\widehat{\eta_\imath}$ be the eigen values of the belongingness, ambiguity and falsity functions respectively, for $i = 1,2,\ldots n$. Their corresponding mean values are $\widehat{\mu_{\widehat{\lambda}}}, \widehat{\mu_{\widehat{\varsigma}}}, \widehat{\mu_{\widehat{\eta}}}$.

Definition 3.3. A spherical fuzzy rough matrix $(SFRM)$ is denoted by $F(S) = \left(\underline{F_{ij}}(S), \overline{F_{ij}}(S)\right)$, has an order of $p \times q$, where $\underline{F_{ij}}(S)$ represents the lower approximation and $\overline{F_{ij}}(S)$ represents the upper approximation of the spherical fuzzy set S. It can be written as: $F(S) = \left(\underline{F_{ij}}(S), \overline{F_{ij}}(S)\right) = \left(\left(\underline{\widehat{\chi_{\imath\jmath}}}(S), \underline{\widehat{\tau_{\imath\jmath}}}(S), \underline{\widehat{\psi_{\imath\jmath}}}(S)\right), \left(\overline{\widehat{\chi_{\imath\jmath}}}(S), \overline{\widehat{\tau_{\imath\jmath}}}(S), \overline{\widehat{\psi_{\imath\jmath}}}(S)\right)\right)_{p \times q}$

where, $\underline{\widehat{\chi_{\imath\jmath}}}(S)$, $\underline{\widehat{\tau_{\imath\jmath}}}(S)$ and $\underline{\widehat{\psi_{\imath\jmath}}}(S)$ are i, j^{th} values of truth, indeterminacy and false mem bership matrices of lower approximation and $\overline{\widehat{\chi_{\imath\jmath}}}(S), \overline{\widehat{\tau_{\imath\jmath}}}(S)$ and $\overline{\widehat{\psi_{\imath\jmath}}}(S)$ are i, j^{th} values of truth, indeterminacy and false membership matrices of upper approximation of spherical fuzzy rough matrix $F(S)$, which satisfy the conditions

$$0 \leq \underline{\widehat{\chi_{\imath\jmath}}}^2(S) + \underline{\widehat{\tau_{\imath\jmath}}}^2(S) + \underline{\widehat{\psi_{\imath\jmath}}}^2(S) \leq 1,$$

$$0 \leq \overline{\widehat{\chi_{\imath\jmath}}}^2(S) + \overline{\widehat{\tau_{\imath\jmath}}}^2(S) + \overline{\widehat{\psi_{\imath\jmath}}}^2(S) \leq 1.$$

Example 3.4. Let us consider a spherical fuzzy rough matrix of order 4 × 2.

$$S = \begin{bmatrix} (0.2, 0.7, 0.01)(0.02, 0.45, 0.30) & (0.51, 0.32, 0.16)(0.18, 0.31, 0.27) \\ (0.19, 0.25, 0.41)(0.52, 0.48, 0.22) & (0.12, 0.23, 0.31)(0.34, 0.29, 0.19) \\ (0.32, 0.24, 0.58)(0.8, 0.15, 0.25) & (0.31, 0.42, 0.38)(0.3, 0.1, 0.5) \\ (0.42, 0.09, 0.3)(0.28, 0.35, 0.17) & (0.4, 0.2, 0.3)(0.2, 0.27, 0.58) \end{bmatrix}$$

Definition 3.5. The Determinant of spherical fuzzy rough matrix $(SFRMS)$ of order $n \times n$, represented as $det(S)$ or $|S|$, has the following definition:

$|S| = \sum_{\sigma \epsilon S_n} \{(\underline{\alpha}_{1\sigma(1)}(\hat{x}), \underline{\beta}_{1\sigma(1)}(\hat{x}), \underline{\gamma}_{1\sigma(1)}(\hat{x})), \ldots, (\underline{\alpha}_{n\sigma(n)}(\hat{x}), \underline{\beta}_{n\sigma(n)}(\hat{x}), \underline{\gamma}_{n\sigma(n)}(\hat{x}))\},$

$$\sum_{\sigma\epsilon S_n}\{(\overline{\alpha}_{1\sigma(1)}(\hat{x}),\overline{\beta}_{1\sigma(1)}(\hat{x}),\overline{\gamma}_{1\sigma(1)}(\hat{x})),\dots,(\overline{\alpha}_{n\sigma(n)}(\hat{x}),\overline{\beta}_{n\sigma(n)}(\hat{x}),\overline{\gamma}_{n\sigma(n)}(\hat{x}))\}$$

$$=\sum_{\sigma\epsilon S_n}\prod_{i=1}^{n}(\underline{s}_{i\sigma(i)},\overline{s}_{i\sigma(i)})$$

where, $i = 1,2,\dots n$.

$\underline{s}_{i\sigma(i)} = \underline{\alpha}_{i\sigma(i)}(\hat{x}), \underline{\beta}_{i\sigma(i)}(\hat{x}), \underline{\gamma}_{i\sigma(i)}(\hat{x})$

$\overline{s}_{i\sigma(i)}) = \overline{\alpha}_{i\sigma(i)}(\hat{x}), \overline{\beta}_{i\sigma(i)}(\hat{x}), \overline{\gamma}_{i\sigma(i)}(\hat{x})$

Each element of $\hat{x}$ is taken from the set S and S_n represents the group consisting of all possible permutations of the set $\{1,2,\dots,n\}$.

Definition 3.6. Let S be $n \times n$ spherical fuzzy rough matrix. Its adjoint is denoted as $adj\ S$ and is defined by $\widehat{A_{\iota\jmath}}$ = $|\ S_{ij}|$where S_{ij} is the transpose of S. The adjoint matrix can thus be represented in a compact form.

$$\widehat{A_{\iota\jmath}} = \sum_{\sigma\epsilon S_{n_i n_j}}\prod_{t\epsilon n_j}((\underline{\hat{\chi}}_{t\sigma(t)},\underline{\hat{\tau}}_{t\sigma(t)},\underline{\hat{\psi}}_{t\sigma(t)}),(\overline{\hat{\chi}}_{t\sigma(t)},\overline{\hat{\tau}}_{t\sigma(t)},\overline{\psi}_{t\sigma(t)}))$$

The set $S_{n_i n_j}$comprises all possible permutations of $n_j$with respect to n_i, where $n_j = \{1,2,\dots,n\}$.

Example 3.7. Let us consider a spherical fuzzy rough matrix of order 2×2.

$$S=\begin{bmatrix}(0.2,0.7,0.01)(0.02,0.45,0.30) & (0.51,0.32,0.16)(0.18,0.31,0.27)\\(0.19,0.25,0.41)(0.52,0.48,0.22) & (0.12,0.23,0.31)(0.34,0.29,0.19)\end{bmatrix}$$

$$|S| = [(0.2,0.7,0.01)(0.02,0.45,0.30).(0.12,0.23,0.31)(0.34,0.29,0.19)] + [(0.19,0.25,0.41)(0.52,0.48,0.22).(0.51,0.32,0.16)(0.18,0.31,0.27)]$$

$= \{[\min(0.2,0.12), \max(0.7,0.23), \max(0.01,0.31)],$

$[min\ (0.02,0.34), max\ (0.45,0.29), max\ (0.30,0.19)]\} +$
$\{[min\ (0.19,0.51), max\ (0.25,0.32), max\ (0.41,0.16)],$

$[min\ (0.52,0.18), max\ (0.48,0.31), max\ (0.22,0.27)]\}$

$= [(0.12,0.7,0.31),(0.02,0.45,0.30)] + [(0.19,0.32,0.41),(0.18,0.48,0.27)]$

$= [\max(0.12,0.19), \min(0.7,0.32), \min(0.31,0.41)],$

$[max(0.02,0.18), min(0.45,0.48), min(0.30,0.27)]$

$= [(0.19,0.32,0.31),(0.18,0.45,0.27)]$

$adj\ S = |S_{ij}| = [(0.19,0.32,0.31),(0.18,0.45,0.27)]$

Definition 3.8. Let $F(S) = [F_{ij}(S)]$ be the spherical fuzzy rough matrix of order $p \times q$ where $[F_{ij}(S)] = \left(\underline{\widehat{\chi_{\iota\jmath}}}(S),\underline{\widehat{\tau_{\iota\jmath}}}(S),\underline{\widehat{\psi_{\iota\jmath}}}(S)\right),\left(\overline{\widehat{\chi_{\iota\jmath}}}(S),\overline{\widehat{\tau_{\iota\jmath}}}(S),\overline{\widehat{\psi_{\iota\jmath}}}(S)\right)$, consequently, the score function can be expressed by

$$\varrho_n = \frac{1}{3}\left(2+\left(\frac{\underline{\underline{\widehat{\chi_{\iota\jmath}}}}+\overline{\overline{\widehat{\chi_{\iota\jmath}}}}}{2}\right)-\left(\frac{\underline{\underline{\widehat{\tau_{\iota\jmath}}}}+\overline{\overline{\widehat{\tau_{\iota\jmath}}}}}{2}\right)-\left(\frac{\underline{\underline{\widehat{\psi_{\iota\jmath}}}}+\overline{\overline{\widehat{\psi_{\iota\jmath}}}}}{2}\right)\right) \longrightarrow (3.1)$$

4. Energy-Based Characterization of Spherical Fuzzy Rough Matrices (SFRM)

Definition 4.1. The spherical fuzzy rough matrix is represented by six matrices of truth $(\underline{\widehat{a_{\iota\jmath}}} \in P(\underline{\widehat{\chi_{\iota\jmath}}}))$, indeterminacy $(\underline{\widehat{b_{\iota\jmath}}} \in P(\underline{\widehat{\tau_{\iota\jmath}}}))$, and falsity $(\underline{\widehat{c_{\iota\jmath}}} \in P(\underline{\widehat{\psi_{\iota\jmath}}}))$ membership values of lower approximations and $(\overline{\widehat{a_{\iota\jmath}}} \in$

$P(\overline{\widehat{\chi_{\imath\jmath}}})$, indeterminacy $(\overline{\widehat{b_{\imath\jmath}}} \in P(\overline{\widehat{\tau_{\imath\jmath}}})$ and falsity $(\overline{\widehat{c_{\imath\jmath}}} \in P(\overline{\widehat{\psi_{\imath\jmath}}})$ membership values of upper approximations. It is denoted as

$$P(S) = \left(P\left(\underline{\widehat{\chi_{\imath\jmath}}}\right), P(\underline{\widehat{\tau_{\imath\jmath}}}), P\left(\underline{\widehat{\psi_{\imath\jmath}}}\right)\right), \left(P\left(\widehat{\overline{\chi}}_{\imath\jmath}\right), P(\widehat{\overline{\tau}}_{\imath\jmath}), P\left(\widehat{\overline{\psi}}_{\imath\jmath}\right)\right)$$

Then the spherical fuzzy rough matrix’s energy is defined as

$$E[P(S)] = \left(\left(E\left[P\left(\underline{\widehat{\chi_{\imath\jmath}}}\right)\right], E[P(\underline{\widehat{\tau_{\imath\jmath}}})], E\left[P\left(\underline{\widehat{\psi_{\imath\jmath}}}\right)\right]\right), \left(E\left[P\left(\widehat{\overline{\chi}}_{\imath\jmath}\right)\right], E[P(\widehat{\overline{\tau}}_{\imath\jmath})], E\left[P\left(\widehat{\overline{\psi}}_{\imath\jmath}\right)\right]\right)\right)$$

$$E[P(S)] = ([\sum_{i=1}^{n} |\underline{\widehat{\lambda}}_{\imath} - \hat{\mu}_{\underline{\widehat{\lambda}}}|, \sum_{i=1}^{n} |\underline{\widehat{\varsigma}}_{\imath} - \hat{\mu}_{\underline{\widehat{\varsigma}}}|, \sum_{i=1}^{n} |\underline{\widehat{\eta}}_{\imath} - \hat{\mu}_{\underline{\widehat{\eta}}}|], [\sum_{i=1}^{n} |\widehat{\overline{\lambda}}_{\imath} - \hat{\mu}_{\widehat{\overline{\lambda}}}|, \sum_{i=1}^{n} |\widehat{\overline{\varsigma}}_{\imath} - \hat{\mu}_{\widehat{\overline{\varsigma}}}|, \sum_{i=1}^{n} |\widehat{\overline{\eta}}_{\imath} - \hat{\mu}_{\widehat{\overline{\eta}}}|])$$

Where $\underline{\widehat{\lambda}}_{\imath}$, $\underline{\widehat{\varsigma}}_{\imath}$ and $\underline{\widehat{\eta}}_{\imath}$ be the eigen values of the belongingness, ambiguity and falsity functions of lower approximation matrices and $\widehat{\overline{\lambda}}_{\imath}$, $\widehat{\overline{\varsigma}}_{\imath}$ and $\widehat{\overline{\eta}}_{\imath}$ be the eigen values of the belongingness, ambiguity and falsity functions of upper approximation matrices respectively, for $i = 1,2, \ldots n$. Their corresponding mean values are $\hat{\mu}_{\underline{\widehat{\lambda}}}$, $\hat{\mu}_{\underline{\widehat{\varsigma}}}$, $\hat{\mu}_{\underline{\widehat{\eta}}}$, $\hat{\mu}_{\widehat{\overline{\lambda}}}$, $\hat{\mu}_{\widehat{\overline{\varsigma}}}$ and $\hat{\mu}_{\widehat{\overline{\eta}}}$ respectively.

Example 4.2. Let S be the Spherical Fuzzy Rough Matrix with the order 2×2.

$$S = \begin{bmatrix} (0.5,0.3,0.1)(0.2,0.1,0.2) & (0.6,0.4,0.1)(0.3,0.2,0.1) \\ (0.4,0.2,0.1)(0.5,0.1,0.1) & (0.7,0.2,0.2)(0.4,0.1,0.2) \end{bmatrix}_{n \times n}$$

S can be expressed as 6 matrices.

$$P(\underline{\widehat{\chi_{\imath\jmath}}}) = \begin{bmatrix} 0.5 & 0.6 \\ 0.4 & 0.7 \end{bmatrix}; P(\underline{\widehat{\tau_{\imath\jmath}}}) = \begin{bmatrix} 0.3 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}; P(\underline{\widehat{\psi_{\imath\jmath}}}) = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

$$P(\widehat{\overline{\chi}}_{\imath\jmath}) = \begin{bmatrix} 0.2 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}; P(\widehat{\overline{\tau}}_{\imath\jmath}) = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}; P(\widehat{\overline{\psi}}_{\imath\jmath}) = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

The eigen values of truth lower matrix $\underline{\widehat{\lambda}}_{\imath} = 1.2856, -0.0856$ and mean $\hat{\mu}_{\underline{\widehat{\lambda}}} = 0.6$. The energy of $P\left(\underline{\widehat{\chi_{\imath\jmath}}}\right)$=1.3712. Similarly find other matrices.

So $E[P(S)] = [(1.3712, 0.5744, 0.3605), (0.8, 0.2828, 0.5292)]$

Theorem 4.3.

Let $P(S)$ be a spherical fuzzy rough matrix. Suppose that $\underline{\widehat{\lambda}}_{\imath}$, $\underline{\widehat{\varsigma}}_{\imath}$, $\underline{\widehat{\eta}}_{\imath}$, $\widehat{\overline{\lambda}}_{\imath}$, $\widehat{\overline{\varsigma}}_{\imath}$ and $\widehat{\overline{\eta}}_{\imath}$ $(i = 1,2, \ldots, n)$ are the eigen values corresponding to the lower and upper approximations of the belongingness matrix $P(\widehat{\chi_{\imath\jmath}})$, hesitancy matrix $P(\widehat{\tau_{\imath\jmath}})$, and falsity matrix $P(\widehat{\psi_{\imath\jmath}})$ respectively.

Then, the following relations hold:

$\sum_{i=1}^{n} |\underline{\widehat{\lambda}}_{\imath} - \hat{\mu}_{\underline{\widehat{\lambda}}}| = \sum_{i=1}^{n} |\underline{\widehat{a_{\imath\imath}}} - \underline{\widehat{a_{\imath\imath}}}| = \sum_{i=1}^{n} |\widehat{\overline{\lambda}}_{\imath} - \hat{\mu}_{\widehat{\overline{\lambda}}}| = \sum_{i=1}^{n} |\widehat{\overline{a}}_{\imath\imath} - \widehat{\overline{a}}_{\imath\imath}| = 0$

$\sum_{i=1}^{n} |\underline{\widehat{\varsigma}}_{\imath} - \hat{\mu}_{\underline{\widehat{\varsigma}}}| = \sum_{i=1}^{n} |\underline{\widehat{b_{\imath\imath}}} - \underline{\widehat{b_{\imath\imath}}}| = \sum_{i=1}^{n} |\widehat{\overline{\varsigma}}_{\imath} - \hat{\mu}_{\widehat{\overline{\varsigma}}}| = \sum_{i=1}^{n} |\widehat{b_{\imath\imath}} - \widehat{\overline{b}}_{\imath\imath}| = 0$ $\sum_{i=1}^{n} |\underline{\widehat{\eta}}_{\imath} - \hat{\mu}_{\underline{\widehat{\eta}}}| = \sum_{i=1}^{n} |\underline{\widehat{c_{\imath\imath}}} - \widehat{c_{\imath\imath}}| = \sum_{i=1}^{n} |\widehat{\overline{\eta}}_{\imath} - \hat{\mu}_{\widehat{\overline{\eta}}}| = \sum_{i=1}^{n} |\widehat{c_{\imath\imath}} - \widehat{\overline{c}}_{\imath\imath}| = 0$

Proof:

Let $P(S)$ be a Spherical fuzzy rough matrix. Then it can be decomposed into three component matrices corresponding to lower and upper approximation of truth, indeterminacy and falsity matrices denoted by $\widehat{\chi_{\imath\jmath}} = (\underline{\widehat{a_{\imath\jmath}}}, \widehat{\overline{a}}_{\imath\jmath})$, $\widehat{\tau_{\imath\jmath}} = (\underline{\widehat{b_{\imath\jmath}}}, \widehat{b_{\imath\jmath}})$ and $\widehat{\psi_{\imath\jmath}} = (\underline{\widehat{c_{\imath\jmath}}}, \widehat{\overline{c}}_{\imath\jmath})$ respectively.

By the definition of $SFRM$, the associated truth, indeterminacy and falsity matrices are diagonal in nature, i.e, all off-diagonal elements are zero.

Since both the lower and upper approximation matrices are diagonal, their eigen values are given by their diagonal elements.

Hence $\underline{\widehat{\lambda}}_\iota = \underline{\widehat{a_{\iota\iota}}}, \underline{\widehat{\varsigma}}_\iota = \underline{\widehat{b_{\iota\iota}}}, \underline{\widehat{\eta}}_\iota = \underline{\widehat{c_{\iota\iota}}},$

$$\widehat{\overline{\lambda}}_\iota = \widehat{\overline{a}_{\iota\iota}}, \widehat{\overline{\varsigma}}_\iota = \widehat{\overline{b}_{\iota\iota}}, \widehat{\overline{\eta}}_\iota = \widehat{\overline{c}_{\iota\iota}}, \qquad i = 1,2,\ldots n.$$

The mean values of the eigen values as

$\hat{\mu}_{\underline{\widehat{\lambda}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{\lambda}}_\iota,\ \hat{\mu}_{\underline{\widehat{\varsigma}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{\varsigma}}_\iota\,,\ \hat{\mu}_{\underline{\widehat{\eta}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{\eta}}_\iota$

$\hat{\mu}_{\underline{\widehat{\lambda}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{a_{\iota\iota}}},\ \hat{\mu}_{\underline{\widehat{\varsigma}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{b_{\iota\iota}}}\,,\ \hat{\mu}_{\underline{\widehat{\eta}}} = \frac{1}{n}\sum_{i=1}^{n} \underline{\widehat{c_{\iota\iota}}}$

Consider the expression, $\sum_{i=1}^{n} |\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}}|$

$$\sum_{i=1}^{n} |\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}}| = \sum_{i=1}^{n} |\underline{\widehat{a_{\iota\iota}}} - \hat{\mu}_{\underline{\widehat{\lambda}}}|$$

Similarly, for upper approximation matrices,

$$\sum_{i=1}^{n} |\widehat{\overline{\lambda}}_\iota - \hat{\mu}_{\widehat{\overline{\lambda}}}| = \sum_{i=1}^{n} |\widehat{\overline{a}_{\iota\iota}} - \hat{\mu}_{\widehat{\overline{\lambda}}}|$$

If all diagonal elements are equal to their corresponding mean values, i.e, $\underline{\widehat{a_{\iota\iota}}} = \hat{\mu}_{\underline{\widehat{\lambda}}}, \underline{\widehat{b_{\iota\iota}}} - \hat{\mu}_{\underline{\widehat{\varsigma}}}, \underline{\widehat{c_{\iota\iota}}} - \hat{\mu}_{\underline{\widehat{\eta}}}, \forall\, i.$

$$\sum_{i=1}^{n} |\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}}| = 0$$

$\therefore \sum_{i=1}^{n} |\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}}| = \sum_{i=1}^{n} |\underline{\widehat{a_{\iota\iota}}} - \underline{\widehat{a_{\iota\iota}}}| = \sum_{i=1}^{n} |\widehat{\overline{\lambda}}_\iota - \hat{\mu}_{\widehat{\overline{\lambda}}}| = \sum_{i=1}^{n} |\widehat{\overline{a}_{\iota\iota}} - \widehat{\overline{a}_{\iota\iota}}| = 0$ $\sum_{i=1}^{n} |\underline{\widehat{\varsigma}}_\iota - \hat{\mu}_{\underline{\widehat{\varsigma}}}| = \sum_{i=1}^{n} |\underline{\widehat{b_{\iota\iota}}} - \underline{\widehat{b_{\iota\iota}}}| = \sum_{i=1}^{n} |\widehat{\overline{\varsigma}}_\iota - \hat{\mu}_{\widehat{\overline{\varsigma}}}| = \sum_{i=1}^{n} |\widehat{b_{\iota\iota}} - \widehat{\overline{b}_{\iota\iota}}| = 0$ $\sum_{i=1}^{n} |\underline{\widehat{\eta}}_\iota - \hat{\mu}_{\underline{\widehat{\eta}}}| = \sum_{i=1}^{n} |\underline{\widehat{c_{\iota\iota}}} - \widehat{c_{\iota\iota}}| = \sum_{i=1}^{n} |\widehat{\overline{\eta}}_\iota - \hat{\mu}_{\widehat{\overline{\eta}}}| = \sum_{i=1}^{n} |\widehat{c_{\iota\iota}} - \widehat{\overline{c}_{\iota\iota}}| = 0$

Hence Proved.

Theorem 4.4.

Let $P(S)$ be a Spherical fuzzy rough matrix. The energy of $P(S)$, determined via the eigen values of its lower and upper approximation matrices of truth, indeterminacy and falsity matrices. Then, the following relations hold:

$\sum_{i=1}^{n}(\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}})^2 = \sum_{i=1}^{n} \underline{\widehat{a_{\iota\iota}}}^2 + \sum_{1\le i<j\le n} \underline{\widehat{a_{\iota\jmath}}}\,\underline{\widehat{a_{\jmath\iota}}} - n\hat{\mu}_{\underline{\widehat{\lambda}}}^2,$ $\sum_{i=1}^{n}(\widehat{\overline{\lambda}}_\iota - \hat{\mu}_{\widehat{\overline{\lambda}}})^2 = \sum_{i=1}^{n} \widehat{\overline{a}_{\iota\iota}}^2 + \sum_{1\le i<j\le n} \widehat{\overline{a}_{\iota\jmath}}\,\widehat{\overline{a}_{\jmath\iota}} - n\hat{\mu}_{\widehat{\overline{\lambda}}}^2,$

$\sum_{i=1}^{n}(\underline{\widehat{\varsigma}}_\iota - \hat{\mu}_{\underline{\widehat{\varsigma}}})^2 = \sum_{i=1}^{n} \underline{\widehat{a_{\iota\iota}}}^2 + \sum_{1\le i<j\le n} \underline{\widehat{a_{\iota\jmath}}}\,\underline{\widehat{a_{\jmath\iota}}} - n\hat{\mu}_{\underline{\widehat{\varsigma}}}^2,$

$\sum_{i=1}^{n}(\widehat{\overline{\varsigma}}_\iota - \hat{\mu}_{\widehat{\overline{\varsigma}}})^2 = \sum_{i=1}^{n} \widehat{\overline{a}_{\iota\iota}}^2 + \sum_{1\le i<j\le n} \widehat{\overline{a}_{\iota\jmath}}\,\widehat{\overline{a}_{\jmath\iota}} - n\hat{\mu}_{\widehat{\overline{\varsigma}}}^2,$

$\sum_{i=1}^{n}(\underline{\widehat{\eta}}_\iota - \hat{\mu}_{\underline{\widehat{\eta}}})^2 = \sum_{i=1}^{n} \underline{\widehat{a_{\iota\iota}}}^2 + \sum_{1\le i<j\le n} \underline{\widehat{a_{\iota\jmath}}}\,\underline{\widehat{a_{\jmath\iota}}} - n\hat{\mu}_{\underline{\widehat{\eta}}}^2,$ $\sum_{i=1}^{n}(\widehat{\overline{\eta}}_\iota - \hat{\mu}_{\widehat{\overline{\eta}}})^2 = \sum_{i=1}^{n} \widehat{\overline{a}_{\iota\iota}}^2 + \sum_{1\le i<j\le n} \widehat{\overline{a}_{\iota\jmath}}\,\widehat{\overline{a}_{\jmath\iota}} - n\hat{\mu}_{\widehat{\overline{\eta}}}^2$

Proof: $\sum_{i=1}^{n}(\underline{\widehat{\lambda}}_\iota - \hat{\mu}_{\underline{\widehat{\lambda}}})^2 = \sum_{i=1}^{n} \underline{\widehat{a_{\iota\iota}}}^2 + \sum_{1\le i<j\le n} \underline{\widehat{a_{\iota\jmath}}}\,\underline{\widehat{a_{\jmath\iota}}} - n\hat{\mu}_{\underline{\widehat{\lambda}}}^2$

Take $L.H.S.$ first, $\sum_{i=1}^{n}(\underline{\widehat{\lambda}}_{i} - \hat{\mu}_{\underline{\widehat{\lambda}}})^2$

Expand the $L.H.S$ in $(\mathrm{a} - \mathrm{b})^2$ formula.

$$\sum_{i=1}^{n}(\underline{\widehat{\lambda}}_{i} - \hat{\mu}_{\underline{\widehat{\lambda}}})^2 = \sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} - 2\,\hat{\mu}_{\underline{\widehat{\lambda}}}\sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i} + \sum_{i=1}^{n}\hat{\mu}_{\underline{\widehat{\lambda}}}^{2} \longrightarrow (1)$$

Substitute $\sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i} = n\hat{\mu}_{\underline{\widehat{\lambda}}}$ in (1)

$\Rightarrow \sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} - 2\,\hat{\mu}_{\underline{\widehat{\lambda}}}n\hat{\mu}_{\underline{\widehat{\lambda}}} + n\hat{\mu}_{\underline{\widehat{\lambda}}}^{\,2}$ $\Rightarrow$
$\sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} - 2n\hat{\mu}_{\underline{\widehat{\lambda}}}^{\,2} + n\hat{\mu}_{\underline{\widehat{\lambda}}}^{\,2}$

$$\Rightarrow \sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} - n\hat{\mu}_{\underline{\widehat{\lambda}}}^{\,2} \longrightarrow (2)$$

Expand $\sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2}$ using trace property.

$$\therefore \sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} = tr(A^2) = \sum_{i=1}^{n}\sum_{i=1}^{n}\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} \longrightarrow (3)$$

Expand the summation

$$\sum_{i=1}^{n}\sum_{i=1}^{n}\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} = \sum_{i=j}\underline{\widehat{a}}_{ii}^{\,2} + \sum_{i\neq j}\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} \longrightarrow (4)$$

$\therefore\ i = j$, $\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} = \underline{\widehat{a}}_{ji}\underline{\widehat{a}}_{ij}$ is equal.

$\therefore\ i \neq j$, counts each pair as twice (once for $i < j$ and $i > j$)

$$\sum_{i\neq j}\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} = 2\sum_{1\leq i<j\leq n}\underline{\widehat{a}}_{ij}\,\underline{\widehat{a}}_{ji} \longrightarrow (5)$$

Subs. (5) in (4)

$$\sum_{i=1}^{n}\sum_{i=1}^{n}\underline{\widehat{a}}_{ij}\underline{\widehat{a}}_{ji} = \sum_{i=j}\underline{\widehat{a}}_{ii}^{\,2} + 2\sum_{1\leq i<j\leq n}\underline{\widehat{a}}_{ij}\,\underline{\widehat{a}}_{ji} \longrightarrow (6)$$

Subs. (6) in (3)

$$\sum_{i=1}^{n}\underline{\widehat{\lambda}}_{i}^{\,2} = \sum_{i=1}^{n}\underline{\widehat{a}}_{ii}^{\,2} + 2\sum_{1\leq i<j\leq n}\underline{\widehat{a}}_{ij}\,\underline{\widehat{a}}_{ji} \longrightarrow (7)$$

Subs. (7) in (2) =
$\sum_{i=1}^{n}\underline{\widehat{a}}_{ii}^{\,2} + +2\sum_{1\leq i<j\leq n}\underline{\widehat{a}}_{ij}\,\underline{\widehat{a}}_{ji} - n\hat{\mu}_{\underline{\widehat{\lambda}}}^{\,2}$ $= R.H.S$

$$\therefore \sum_{i=1}^{n}(\underline{\widehat{\lambda}}_{i} - \hat{\mu}_{\underline{\widehat{\lambda}}})^2 = \sum_{i=1}^{n}\underline{\widehat{a}}_{ii}^{\,2} + \sum_{1\leq i<j\leq n}\underline{\widehat{a}}_{ij}\,\underline{\widehat{a}}_{ji} - n\hat{\mu}_{\underline{\widehat{\lambda}}}^{2}$$

Similarly, we prove lower and upper approximations of indeterminacy and falsity matrices.

5. MCDM Framework Employing Energy Rankings of SFRM’s

This section focuses on a decision-making approach is presented to determine the most suitable alternative among several choices by utilizing the energy of the Spherical fuzzy rough matrix. Consider a decision-making problem involving o potential alternatives that are assessed based on m evaluation criteria. The assessment process is carried out by a panel of n experts. The sets of alternatives, criteria and experts are represented respectively as $P = \{P_1, P_2, \dots, P_O\}$, $R = \{R_1, R_2, \dots, R_m\}$ and $EX = \{EX_1, EX_2, \dots, EX_n\}$.

Step 1: Each expert assign weights to the criteria and evaluate every alternative according to each criterion. The weight and rating information provided by the experts are represented in matrix form for computational convenience. Assume that the experts evaluations of them criteria formulated as a weight matrix W of order $m \times n$.

$$\begin{array}{c c} & \begin{array}{cccc} EX_1 & EX_2 & \dots & EX_n \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_m \end{array} & \begin{bmatrix} (\alpha_{11},\beta_{11},\gamma_{11}) & (\alpha_{12},\beta_{12},\gamma_{12}) & \cdots & (\alpha_{1n},\beta_{1n},\gamma_{1n}) \\ (\alpha_{21},\beta_{21},\gamma_{21}) & (\alpha_{22},\beta_{22},\gamma_{22}) & \cdots & (\alpha_{2n},\beta_{2n},\gamma_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha_{m1},\beta_{m1},\gamma_{m1}) & (\alpha_{m2},\beta_{m2},\gamma_{m2}) & \cdots & (\alpha_{mn},\beta_{mn},\gamma_{mn}) \end{bmatrix} \end{array}$$

Assume the expert evaluations for alternative D_1 are represented by a matrix of order $n \times m$ for the alternative D_1.

$$\begin{array}{c c} & \begin{array}{cccc} R_1 & R_2 & \dots & R_m \end{array} \\ \begin{array}{c} EX_1 \\ EX_2 \\ \vdots \\ EX_n \end{array} & \begin{bmatrix} (\mu_{11},\vartheta_{11},\delta_{11}) & (\mu_{12},\vartheta_{12},\delta_{12}) & \cdots & (\mu_{1n},\vartheta_{1n},\delta_{1n}) \\ (\mu_{21},\vartheta_{21},\delta_{21}) & (\mu_{22},\vartheta_{22},\delta_{22}) & \cdots & (\mu_{2n},\vartheta_{2n},\delta_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{n1},\vartheta_{n1},\delta_{n1}) & (\mu_{n2},\vartheta_{n2},\delta_{n2}) & \cdots & (\mu_{nm},\vartheta_{nm},\delta_{nm}) \end{bmatrix} \end{array}$$

Step 2: Determine the weights of the experts. Suppose $EX_1, EX_2, \dots, EX_n$ represent the experts, each associated with an individual weight.

Let $EX_1 = (e_1, f_1, g_1)$, $EX_2 = (e_2, f_2, g_2), \dots, EX_n = (e_n, f_n, g_n)$

Step 3: Define the matrix based on the $SFRS$ for both criteria and alternatives. The spherical fuzzy rough matrix corresponding to the criteria is formulated to illustrate the connection between each criterion and the weights determined by experts.

$$\begin{aligned} W(R_1EX_1) &= [(\min(e_1,\alpha_{11}), \max(f_1,\beta_{11}), \max(g_1,\gamma_{11})), \\ &\qquad (max(e_1,\alpha_{11}), min(f_1,\beta_{11}), min(g_1,\gamma_{11}))] \\ &= [\left(\underline{\alpha_{11}}, \underline{\beta_{11}}, \underline{\gamma_{11}}\right), (\overline{\alpha_{11}}, \overline{\beta_{11}}, \overline{\gamma_{11}})] \end{aligned}$$

$$W = \begin{array}{c c} & \begin{array}{ccc} EX_1 & \cdots & EX_n \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_m \end{array} & \begin{bmatrix} [\left(\underline{\alpha_{11}}, \underline{\beta_{11}}, \underline{\gamma_{11}}\right), (\overline{\alpha_{11}}, \overline{\beta_{11}}, \overline{\gamma_{11}})] & \cdots & [\left(\underline{\alpha_{1n}}, \underline{\beta_{1n}}, \underline{\gamma_{1n}}\right), (\overline{\alpha_{1n}}, \overline{\beta_{1n}}, \overline{\gamma_{1n}})] \\ [\left(\underline{\alpha_{21}}, \underline{\beta_{21}}, \underline{\gamma_{21}}\right), (\overline{\alpha_{21}}, \overline{\beta_{21}}, \overline{\gamma_{21}})] & \cdots & [\left(\underline{\alpha_{2n}}, \underline{\beta_{2n}}, \underline{\gamma_{2n}}\right), (\overline{\alpha_{2n}}, \overline{\beta_{2n}}, \overline{\gamma_{2n}})] \\ \vdots & \vdots & \vdots \\ [\left(\underline{\alpha_{m1}}, \underline{\beta_{m1}}, \underline{\gamma_{m1}}\right), (\overline{\alpha_{m1}}, \overline{\beta_{m1}}, \overline{\gamma_{m1}})] & \cdots & [\left(\underline{\alpha_{mn}}, \underline{\beta_{mn}}, \underline{\gamma_{mn}}\right), (\overline{\alpha_{mn}}, \overline{\beta_{mn}}, \overline{\gamma_{mn}})] \end{bmatrix} \end{array}$$

For the alternatives, $SFRM$ is constructed to illustrate the link between each criterion and the alternatives.

$$P_1(EX_1R_1) = \left(\min(\alpha_{11},\mu_{11}), max(\beta_{11},\vartheta_{11}), max(\gamma_{11},\delta_{11})\right),$$

$$\left(\max(\alpha_{11},\mu_{11}), min(\beta_{11},\vartheta_{11}), min(\gamma_{11},\delta_{11})\right) = (\left(\underline{\mu_{11}}, \underline{\vartheta_{11}}, \underline{\delta_{11}}\right), (\overline{\mu_{11}}, \overline{\vartheta_{11}}, \overline{\delta_{11}}))$$

$$\begin{array}{ccc} R_1 & \cdots & R_m \end{array}$$

$$P_1 = \begin{array}{c} EX_1 \\ EX_2 \\ \vdots \\ EX_n \end{array} \begin{bmatrix} [(\underline{\mu_{11}}, \underline{\vartheta_{11}}, \underline{\delta_{11}}), (\overline{\mu_{11}}, \overline{\vartheta_{11}}, \overline{\delta_{11}})] & \cdots & [(\underline{\mu_{1m}}, \underline{\vartheta_{1m}}, \underline{\delta_{1m}}), (\overline{\mu_{1m}}, \overline{\vartheta_{1m}}, \overline{\delta_{1m}})] \\ [(\underline{\mu_{21}}, \underline{\vartheta_{21}}, \underline{\delta_{21}}), (\overline{\mu_{21}}, \overline{\vartheta_{21}}, \overline{\delta_{21}})] & \cdots & [(\underline{\mu_{2m}}, \underline{\vartheta_{2m}}, \underline{\delta_{2m}}), (\overline{\mu_{2m}}, \overline{\vartheta_{2m}}, \overline{\delta_{2m}})] \\ \vdots & \vdots & \vdots \\ [(\underline{\mu_{n1}}, \underline{\vartheta_{n1}}, \underline{\delta_{n1}}), (\overline{\mu_{n1}}, \overline{\vartheta_{n1}}, \overline{\delta_{n1}})] & \cdots & [(\underline{\mu_{nm}}, \underline{\vartheta_{nm}}, \underline{\delta_{nm}}), (\overline{\mu_{nm}}, \overline{\vartheta_{nm}}, \overline{\delta_{nm}})] \end{bmatrix}$$

Step 4: During this stage, the non-square matrix is adapted into a square matrix. The matrix W is further divided into six matrices, representing the lower and upper approximations of truth, indeterminacy and falsity matrices, collectively denoted as

$(W(\underline{\widehat{\chi_{ij}}}), W(\underline{\widehat{\tau_{ij}}}), W(\underline{\widehat{\psi_{ij}}}))$ and $(W(\overline{\widehat{\chi_{ij}}}), W(\overline{\widehat{\tau_{ij}}}), W(\overline{\widehat{\psi_{ij}}}))$. Similarly, P_1 matrix is represented as $(P_1(\underline{\widehat{\chi_{ij}}}), P_1(\underline{\widehat{\tau_{ij}}}), P_1(\underline{\widehat{\psi_{ij}}}))$ and $(P_1(\overline{\widehat{\chi_{ij}}}), P_1(\overline{\widehat{\tau_{ij}}}), P_1(\overline{\widehat{\psi_{ij}}}))$.

$$P_1(\underline{\widehat{\chi_{ij}}})_{n\times m} \times W(\underline{\widehat{\chi_{ij}}})_{m\times n} = P_1(\underline{\hat{\chi}}) = \begin{bmatrix} \underline{\mu\alpha_{11}} & \underline{\mu\alpha_{12}} & \dots & \underline{\mu\alpha_{1n}} \\ \underline{\mu\alpha_{21}} & \underline{\mu\alpha_{22}} & \dots & \underline{\mu\alpha_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{\mu\alpha_{n1}} & \underline{\mu\alpha_{n2}} & \dots & \underline{\mu\alpha_{nn}} \end{bmatrix}_{n\times n}$$

Step 5: Matrix energy is evaluated through the concept of a $SFRM$. For every alternative, six energy values are obtained, corresponding to the lower and upper ap proximations of belongingness, ambiguity and falsity matrices.

$$E(P_1) = ((E[P_1(\underline{\hat{\chi}})], E[P_1(\underline{\hat{\tau}})], E[P_1(\underline{\hat{\psi}})]), (E[P_1(\overline{\hat{\chi}})], E[P_1(\overline{\hat{\tau}})],, E[P_1(\overline{\hat{\psi}})]))$$

Step 6: This process is repeated for all k possible alternatives, and the approximate $SFRM$ energies $E(P_1), E(P_2), \dots, E(P_O)$ are computed for every alternative.

Step 7: Ranking values are derived by evaluating the score function for every $SFRM$ energy and alternatives are then ordered from highest to lowest.

6. Numerical Example

Let us assume that problem of selecting good steel rod for construction work performing well across all criteria. Six companies are selected as alternatives $(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \widehat{A}_4, \widehat{A}_5, \widehat{A}_6)$, the criteria are:

R_1=Strength and hardness,

R_2=Ductility,

R_3=Corrosion resistance.

Three decision makers are involved in assessing the problem (EX_1, EX_2, EX_3) who are experts in selecting a best steel rod from the best company. Table.1 provides the linguistic variables for spherical fuzzy numbers, which help experts assess the alternatives.

Table1. Linguistic variables for spherical fuzzy numbers

S.No	Linguistic variable (Code)	Spherical fuzzy numbers
1.	Low quality	(0.25,0.1,0.2)
2.	Medium quality	(0.45,0.5,0.51)
3.	Best quality	(0.8,0.3,0.5)

Table2. Weights of criteria

Criteria	EX_1	EX_2	EX_3
R_1	$\widehat{B_{\jmath}}$	$\widehat{B_{\jmath}}$	$\widehat{M_{\jmath}}$
R_2	$\widehat{B_{\jmath}}$	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$
R_3	$\widehat{M_{\jmath}}$	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$

Step 1: Tables 2 and 3 display the expert-assigned importance levels for the criteria and the evaluated alternatives.

Step 2: The weight of experts:

$$EX_1 = \widehat{B_{\jmath}} = (0.8,0.3,0.5);\ EX_2 = \widehat{M_{\jmath}} = (0.45,0.5,0.51);\ EX_3 = \widehat{M_{\jmath}} = (0.45,0.5,0.51)$$

Step 3: From the obtained values, $SFRM's$ are developed for both the criteria and the alternatives. The $SFRM$ corresponding to the criteria describes the association between each criterion and the expert-assigned weights, as reported in Table 4. In a similar manner, the $SFRM$ for the alternatives reflects the linkage between the criteria and the alternatives as presented in Table 5.

Step 4: The product of the lower truth matrices of alternatives and the weights assigned to the criteria.

$$\widehat{A_{\jmath 1}}\left(\underline{\hat{\chi}}\right) = \widehat{A_{\jmath 1}}(\underline{\widehat{\chi_{\iota\jmath}}})_{y\times x} \times W(\underline{\widehat{\chi_{\iota\jmath}}})_{x\times y} = \begin{bmatrix} 0.8 & 0.45 & 0.45 \\ 0.45 & 0.45 & 0.25 \\ 0.45 & 0.45 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.8 & 0.45 & 0.45 \\ 0.8 & 0.45 & 0.45 \\ 0.45 & 0.45 & 0.45 \end{bmatrix}$$

$$\widehat{A_{\jmath 1}}\left(\underline{\hat{\chi}}\right) = \begin{bmatrix} 1.2025 & 0.7650 & 0.7650 \\ 0.8325 & 0.5175 & 0.5175 \\ 1.0800 & 0.7650 & 0.7650 \end{bmatrix}$$

$\widehat{\lambda_\iota}$ =2.4527, 0.0323 and 0. Truth lower approximation matrix energy = 3.2488

Step 5: The energy levels of both the lower and upper approximation matrices for belongingness, ambiguity and falsity are determined.

Table 3. Scores assigned to each alternative.

Companies	Experts	R_1	R_2	R_3
$\widehat{A_{\jmath 1}}$	EX_1	$\widehat{B_{\jmath}}$	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$
	EX_2	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$	$\widehat{L_{\jmath}}$
	EX_3	$\widehat{B_{\jmath}}$	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$
$\widehat{A_{\jmath 2}}$	EX_1	$\widehat{M_{\jmath}}$	$\widehat{L_{\jmath}}$	$\widehat{B_{\jmath}}$
	EX_2	$\widehat{L_{\jmath}}$	$\widehat{B_{\jmath}}$	$\widehat{B_{\jmath}}$
	EX_3	$\widehat{B_{\jmath}}$	$\widehat{B_{\jmath}}$	$\widehat{M_{\jmath}}$
$\widehat{A_{\jmath 3}}$	EX_1	$\widehat{M_{\jmath}}$	$\widehat{B_{\jmath}}$	$\widehat{B_{\jmath}}$

	EX_2 EX_3	$\widehat{B_{\lambda}}$ $\widehat{L_{\lambda}}$	$\widehat{L_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{B_{\lambda}}$ $\widehat{B_{\lambda}}$
$\widehat{A_{\lambda}}_{4}$	EX_1 EX_2 EX_3	$\widehat{B_{\lambda}}$ $\widehat{L_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{B_{\lambda}}$ $\widehat{M_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{M_{\lambda}}$ $\widehat{B_{\lambda}}$ $\widehat{L_{\lambda}}$
$\widehat{A_{\lambda}}_{5}$	EX_1 EX_2 EX_3	$\widehat{L_{\lambda}}$ $\widehat{B_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{M_{\lambda}}$ $\widehat{M_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{M_{\lambda}}$ $\widehat{B_{\lambda}}$ $\widehat{M_{\lambda}}$
$\widehat{A_{\lambda}}_{6}$	EX_1 EX_2 EX_3	$\widehat{L_{\lambda}}$ $\widehat{B_{\lambda}}$ $\widehat{M_{\lambda}}$	$\widehat{M_{\lambda}}$ $\widehat{L_{\lambda}}$ $\widehat{B_{\lambda}}$	$\widehat{B_{\lambda}}$ $\widehat{M_{\lambda}}$ $\widehat{L_{\lambda}}$

Table 4. Spherical fuzzy rough matrix for criteria.

Criteria	EX_1	EX_2	EX_3
R_1	[(0.8,0.3,0.5), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.45,0.5,0.51)]
R_2	[(0.8,0.3,0.5), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.45,0.5,0.51)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]
R_3	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.45,0.5,0.51)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]

Table 5. Spherical fuzzy rough matrix for alternative.

$\widehat{A_{\lambda}}_{1}$	R_1	R_2	R_3
EX_1	[(0.8,0.3,0.5), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]
EX_2	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.25,0.5,0.51), (0.45,0.1,0.2)]
EX_3	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.45,0.5,0.51), (0.8,0.3,0.5)]	[(0.8,0.3,0.5), (0.8,0.3,0.5)]

Energy of

$$\widehat{A_{\gimel 1}} = [(3.2488,2.4815,3.0941),(6.2649,1.2159,2.8182)].$$

Table 6. Ranking order of alternatives.

Company	Score function	Ranking
$\widehat{A_{\gimel 1}}$	0.6507	3
$\widehat{A_{\gimel 2}}$	0.6886	2
$\widehat{A_{\gimel 3}}$	0.5353	6
$\widehat{A_{\gimel 4}}$	0.7192	1
$\widehat{A_{\gimel 5}}$	0.5448	5
$\widehat{A_{\gimel 6}}$	0.5907	4

Similarly, the energy is computed for each alternatives.

Step 6: The alternatives are evaluated by computing their respective energy scores.

$\widehat{A_{\gimel 2}}$ = [(2.7645,2.3867,3.0873), (6.5600,1.1030,2.6158)],

$\widehat{A_{\gimel 3}}$ = [(3.5405,3.4719,3.0805), (5.8950,1.0533,2.6180)],

$\widehat{A_{\gimel 4}}$= [(3.5379,2.1335,3.0739), (5.8861,1.2737,2.6275)],

$\widehat{A_{\gimel 5}}$= [(3.1237,2.3867,3.0873), (5.8861,1.4333,2.8335)],

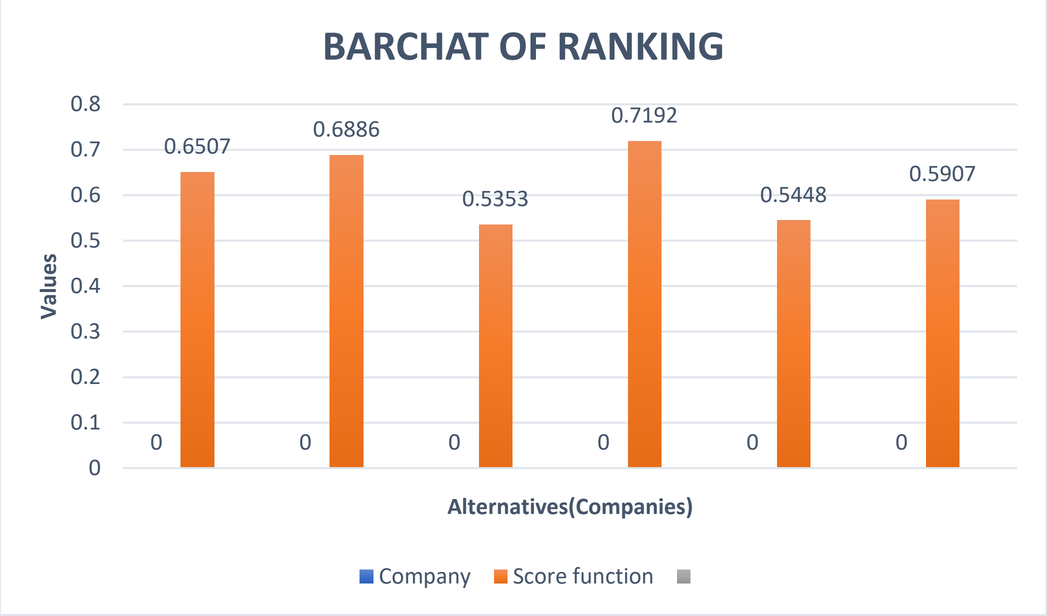
$\widehat{A_{\gimel 6}}$ = [(2.8965,2.2343,3.0805), (5.6050,1.2133,2.4294)].

Step 7: The values of the score function are obtained from the $SFRM$ energy. Equation 3.1 is applied to obtain the score function of $\widehat{A_{\gimel 1}}$ and this procedure is repeated for all other alternatives.

Score function of $\widehat{A_{\gimel 1}} = \frac{1}{3}\left[2 + \left[\frac{3.2488+6.2649}{2}\right] - \left[\frac{2.4815+1.2159}{2}\right] - \left[\frac{3.0941+2.8182}{2}\right]\right]$

$P(\widehat{A_{\gimel 1}})$ = 0.6507.

The resulting rank arrangement is shown below: $\widehat{A_{\gimel 4}}$, $\widehat{A_{\gimel 2}}$, $\widehat{A_{\gimel 1}}$, $\widehat{A_{\gimel 6}}$, $\widehat{A_{\gimel 5}}$, $\widehat{A_{\gimel 3}}$. Table 6 shows that alternative 2 achieves the highest performance compared to the others. Con sequently, company 2 is selected as the preferred company in that region. Figure 1 illustrates the score function values of all alternatives in a barchat.



Conclusion

This paper presents new concepts of spherical fuzzy determinant, spherical fuzzy rough determinant and spherical fuzzy rough adjoint. These definitions extend classical matrix theory to a spherical fuzzy rough framework. The concept of energy of a spherical fuzzy matrix was also introduced. The proposed energy measure helps analyze matrix behaviour under uncertainity. An $MCDM$ application was carried out to demonstrate practical usefulness. The results show that the proposed approach provides consistent and reliable rankings. The integration of spherical fuzzy and rough structures improves decision accuracy. The developed framework can handle complex and vague information effectively. This study opens new directions for further research in fuzzy matrix theory and decision-making models.

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