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Coexistence and Stability Analysis of a Harvested Tri-Trophic Ecological Model with Hybrid Holling Responses and Intra-Specific Competition

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Abstract:- This paper presents and analyzes a three-species ecological model incorporating Holling type I and Holling type II functional responses together with intra-specific competition. The model describes nonlinear interactions among the species and includes density-dependent growth of the prey population. The positivity and boundedness of the solutions are established to ensure the biological feasibility of the system. The existence of equilibrium points is investigated, and the local stability of the coexistence equilibrium point is analyzed using the Routh–Hurwitz criterion. It is shown that the system is locally asymptotically stable and globally stable under suitable parametric conditions. Furthermore, the effect of intra-specific competition on the system dynamics is examined, demonstrating that it prevents unbounded growth and promotes species coexistence. Numerical simulations are performed to validate the analytical results. The time-series plots and phase portraits illustrate the stability characteristics of the system and exhibit various dynamical behaviors, including coexistence and species extinction under parameter variations. The findings provide significant insights into complex ecological interactions and emphasize the important role of competition in maintaining system stability.

Keywords: Steady states; Three-species food-chain model; Harvesting; Local stability analysis; Global stability analysis.

1. Introduction

Mathematical ecology plays an important role in understanding the dynamics of interacting populations and their long-term behavior. Population growth and species persistence are strongly influenced by ecological mechanisms such as competition, predation, diffusion, and the Allee effect. In particular, the Allee effect, which describes reduced population growth at low population density, has attracted considerable attention due to its influence on extinction thresholds and coexistence patterns [6, 7]. The interaction between predation and competition is another fundamental aspect in ecological systems, as it significantly affects species diversity and stability. Intraguild predation, where competing species may also prey upon one another, has been recognized as an important ecological process that governs coexistence and competitive exclusion [2, 3]. Moreover, nonconsumptive effects among competing species further contribute to the complexity of ecological interactions [1]. Spatial heterogeneity and species dispersal also play a crucial role in determining population dynamics. Diffusion-based mathematical models are widely used to study the movement of populations in heterogeneous environments. Recent studies have shown that environmental variation significantly influences carrying capacity and long-term persistence of species [?]. Such spatial effects are essential for understanding realistic ecological System. Motivated by these studies, the present work focuses on the analysis of a

nonlinear population model incorporating ecological interactions and spatial effects. The stability properties and qualitative behavior of the system are investigated to understand the conditions for persistence, coexistence, and asymptotic stability of the equilibrium points [17-20]. Predator-prey interactions constitute one of the most fundamental topics in mathematical ecology, as they describe the coexistence and long-term behavior of biological populations. In recent years, considerable attention has been devoted to the study of predator–prey systems involving complex ecological mechanisms such as cannibalism, disease transmission, prey defense, and diffusion effects [8, 9, 10, 11]. In particular, prey cannibalism has been shown to significantly alter the stability and persistence of ecological systems by modifying growth and predation dynamics [8]. Fractionalorder differential equations have gained considerable attention as an effective mathematical framework for capturing memory and hereditary characteristics in biological systems. In contrast to Fractional-order systems offer a more

precise representation compared to classical integer-order models.and realistic description of the population dynamics and their temporal evolution [13,14]. The study of stability in prey–predator systems with mixed functional responses has become a significant topic in mathematical ecology, These models provide a more effective description of complex interactions among species. Inspired by earlier research, this study examines the dynamical behavior of a fractional-order predator–prey system that includes nonlinear ecological interactions. [15] The qualitative characteristics of the proposed model are investigated with particular emphasis on the Study of the existence and local stability of equilibrium points, together with the conditions for global stability are also

derived under appropriate assumptions. Furthermore, the role of system parameters in determining the stability and persistence of species is examined in detail. To support the theoretical findings, numerical simulations are performed. The results illustrate the dynamical behavior of the system and confirm the analytical outcomes, including stability, coexistence, and possible extinction scenarios under varying parameter values. The study provides valuable insights into the impact of memory effects and nonlinear interactions on ecological system dynamics.

2. Formulation of the Proposed System

In this section, an ecological model consisting of three species in a food chain is formulated. The prey–intermediate predator interaction is governed by a Holling type I response, while the secondary–top predator interaction is modeled using a Holling type II response. Assume that $G_1(t)$, $G_2(t)$, G_3 (t) denote the population densities of the prey, seondary predator, and top predator, respectively, for all t > 0. The spredation processes among the species are governed by these functional responses, which effectively capture the feeding behavior and Sinteraction dynamics within the system. Furthermore, harvesting and natural mortality and intraspecific are incorporated into the predator populations to account for external and environmental influences. That the prey population that $G_1(t)$ is assumed to follow logistic growth with intrinsic growth rate $\widetilde{C}$ and carrying capacity e.

$$\frac{dG_1}{dt} = \tilde{C}G_1 \ (1-\frac{G_1}{e}) - \frac{\widetilde{\alpha_1}G_1G_2}{(1+G_1\bar{S})} - \widetilde{\eta_1}G_1$$

$$\frac{dG_2}{dt} = \widehat{K_1}\frac{\widetilde{\alpha_1}G_1G_2}{(1+G_1\bar{S})} - \widehat{\delta}_1G_2 - \widehat{\alpha_2}G_2G_3 - \widetilde{\gamma_1}\ G_2^2 - \widetilde{\eta_2}G_2 \quad (2.1)$$

$$\frac{dG_2}{dt} = \ \widehat{K_2}\,\widehat{\alpha_2}G_2G_3 - \widehat{\delta_2}G_3 - \widetilde{\gamma_2}G_3^2$$

All intial population values are non-negative,G_i (0) ≥ 0, i = 1,2,3.

The interaction term $\frac{\widetilde{\alpha_1}G_1G_2}{1+\bar{s}G_1}$ is characterized by a Holling type II functional response, illustrating the predatory interaction with the prey population G_1 by the secondary predator G_2. Furthermore, the terms $\widetilde{\eta_1}G_1$ and describe $\widetilde{\eta_2}G_2$ the harvesting effects applied to the prey and secondary predator populations, respectively.The secondary predator G_2 gains energy from consuming prey with conversion efficiency $\widehat{K_1}$, loses due to natural death at rate $\widehat{\delta_1}$,and is further reduced due to predation by the top predator G_3. In addition, the secondary predator experiences intraspecific competition represented by the term $\widetilde{\gamma_1}G_2^2$ .The top predator G_3 increases by consuming the secondary predator G_2 with predation rate $\widehat{\alpha_2}$. The top predator G_3 gains energy from consuming secondary predator with conversion efficiencys $\widehat{K_2}$ It decreases due to natural mortality at rate $\widehat{\delta_2}$ and is further regulated by intraspecific competition represented by the term $\widetilde{\gamma_2}G_3^2$. No harvesting is considered for the top predator G_2 in this model.

Table 1: Description of parameters used in the model.

Parameter	Description
$\widetilde{C}$	Intrinsic growth rate of the prey population
e	Environmental carrying capacity of the prey population
$\widehat{\alpha_1}$	Rate of predation exerted by the secondary predator on the prey population
$\widehat{\alpha_2}$	Rate of predation exerted by the top predator on the secondary predator population
$\bar{S_1}$	Half-saturation constant
$\widetilde{\eta_1}$	Harvesting rate of prey population
$\widetilde{\eta_2}$	Harvesting rate of secondary predator population
$\hat{\delta}_1$	Mortality rate of the secondary predator
$\widehat{\delta_2}$	Mortality rate of the top predator
$\widehat{K_1}$	Conversion efficiency of prey into secondary predator on prey
$\widehat{K_2}$	Conversion efficiency of secondary predator into on top predator
$\widetilde{\gamma_1}$	Intraspecific competition coefficient of secondary predator on prey
$\widetilde{\gamma_2}$	Intraspecific competition coefficient of top predator

3. Examination of the System for Positivity and Boundedness Properties

Theorem 3.1.
If it is non-negative for all 't', then all solutions of the system (2.1).

Proof: By integreting each equation of the system (2.1) provide the following.

$$G_1(t) = G_1(0) \exp\left(\int \left[\tilde{C}\left(1-\frac{G_1}{e}\right) - \frac{\widetilde{\alpha_1} G_1}{(1+G_1\bar{S})} - \widetilde{\eta_1}\right] dt\right),$$

$$G_2(t) = G_2(0) \exp\left(\int \left[\widehat{K_1}\frac{\widetilde{\alpha_1} G_2}{(1+G_1\bar{S})} - \hat{\delta}_1 - \widehat{\alpha_2} G_3 - \widetilde{\gamma_1}\, G_2^2 - \widetilde{\eta_2}\right] dt\right),$$

$$G_3(t) = G_3(0) \exp\left(\int \left[\widehat{K_2}\,\widehat{\alpha_2} G_2 - \widehat{\delta_2} - \widetilde{\gamma_2} G_3\right] dt\right)$$

Since $G_1(0) \geq 0,\ G_2(0) \geq 0,\ G_3(0) \geq 0$ it follows that $G_1(t) \geq 0,\ G_1(t) \geq 0,\ G_1(t) \geq 0$ for all $t \geq 0$.Hence, the solutions of the system remain non-negative for all time $t \geq 0$.

Theorem 3.2.
The solutions of system (2.1) remain uniformly bounded in R^3_+.

Proof: Define the function $\zeta = G_1 + G_2 + G_3$.

$$\frac{d\zeta}{dt} = \frac{dG_1}{dt} + \frac{dG_2}{dt} + \frac{dG_3}{dt}.$$

Using system (2.1), we obtain

$$\frac{d\zeta}{dt} = \left[\tilde{C}\left(1-\frac{G_1}{e}\right) - \breve{\eta}_1\right] G_1 - \left[\hat{\delta}_1 - \breve{\eta}_2\right] G_2 - \hat{\delta}_2 G_3.$$

For any positive constant $\Phi > 0$, we obtain

$$\frac{d\zeta}{dt} + \Phi\zeta = \left[\tilde{C}\left(1-\frac{G_1}{e}\right) + \Phi - \breve{\eta}_1\right]G_1 - [\hat{\delta}_1 - \breve{\eta}_2 + \Phi]G_2 - [\hat{\delta}_2+\Phi]G_3.$$

If we choose $\Phi = \hat{\delta}_2$, then

$$\frac{d\zeta}{dt} + \Phi\zeta \le \left[\frac{e(\tilde{C}-\breve{\eta}_1)+\Phi}{\tilde{C}}\right] + \left[\frac{1}{\hat{\delta}_2-\breve{\eta}_2+\Phi}\right] \le \varrho$$

Where $\varrho = \left[\frac{e(\tilde{C}-\breve{\eta}_1)+\Phi}{\tilde{C}}\right] + \left[\frac{1}{\hat{\delta}_2-\breve{\eta}_2+\Phi}\right].$

Assume that Φ be an absolutely continuous function that satisfies the differential inequalitywhere $(\alpha_1, \alpha_2) \in \mathbb{R}^3_+,\ \alpha_1 \neq 0$. Then, for all $t \ge T \ge 0$,

$$\phi(t) \le \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1} - \phi(T)\right)e^{-\alpha_1(t-T)}.$$

Using the above conditions, we obtain

$$0 \le \zeta(G_1, G_2, G_3) \le \frac{\dot{\varrho}}{\Phi}(1-e^{-\Phi t}) + \zeta(G_1(0), G_2(0), G_3(0))e^{-\Phi t}.$$

Taking the limit as $t \to \infty$, we obtain

$$0 \le \zeta(G_1, G_2, G_3) \le \frac{\dot{\varrho}}{\Phi}.$$

Therefore, all solutions of system (2.1) that start in R 3 + remain bounded and ultimately enter the region

$\Gamma = \{(G1, G2, G3) \in R^3_+ : 0 \le \zeta \le \frac{\varrho}{\Phi} + \varepsilon\}, \forall\, \varepsilon > 0.$

Hence, the system (2.1) is uniformly bounded.[18]

5.Analytical Study of the System

The system exhibits four equilibrium points.

Table 2 summarizes the equilibrium points of system (2.1) together with the corresponding conditions required for their existence.

Equilibrium points of system (2.1) corresponding to the given conditions

S.No	**Equilibrium points**	**Biological significance**
1.	$E_0\ (0,0,0)$	Trivial
2	$E_1\ (\frac{e(\tilde{C}-\breve{\eta}_1)}{\tilde{C}}, 0,0)$	Prey only present

3.	$E_2(G_1^*, G_2^*, 0)$	Prey-secondary predator present
4.	$E_3(G_1^*, G_2^*, G_3^*)$	Coexistence

A steady-state solution of the model $E_2(G_1^*, G_2^*, 0)$ exists if

$$G_2^* = (1 + s_1 G_1^*)\left(1 - \frac{G_1^*}{e}\right)\left(\frac{\tilde{c}}{\breve{\alpha}_1}\right),$$

where G_1^* is a real root of the cubic equation

$$B_3 G_1^{*3} + B_2 G_1^{*2} + B_1 G_1^{*1} + B_0 G_1^{*0} = 0.$$

Here, B_0, B_1, B_2, B_3 are constants depending on the parameters $\tilde{C}, e, \breve{\alpha}_1, \bar{S}_1, \breve{\eta}_1, \widehat{K}_1, \tilde{\gamma}_2, \breve{\eta}_2, \hat{\delta}_1, \hat{\delta}_2, \breve{\alpha}_2$ and $\widehat{K}_2$. The interior equilibrium point $E_3(G_1^*, G_2^*, G_3^*)$ is given by

$$G_3^* = \frac{\widehat{K}_2 \breve{\alpha}_2 G_2^* - \hat{\delta}_2}{\tilde{\gamma}_2}$$

where G_1^* satisfies the same cubic equation

$$B_3 G_1^{*3} + B_2 G_1^{*2} + B_1 G_1^{*1} + B_0 G_1^{*0} = 0.$$

Thus, both E_2 and E_3 are governed by a common cubic polynomial equation.

5. Dynamics of the System in the Vicinity of Equilibrium Points

Evaluating the Jacobian matrix at the equilibrium point $E(G_1, G_2, G_3)$, we obtain

$$J = \begin{bmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} & 0 \\ \widetilde{K}_{21} & \widetilde{K}_{22} & \widetilde{K}_{23} \\ 0 & \widetilde{K}_{32} & \widetilde{K}_{33} \end{bmatrix} \tag{5.2}$$

Here,

$$\widetilde{K}_{11} = \tilde{C} - \frac{2\tilde{C}G_1}{e} - \frac{\breve{\alpha}_1 G_2}{(1+sG_1)^2} - \breve{\eta}_1, \qquad \widetilde{K}_{12} = -\frac{\breve{\alpha}_1 G_1}{1+sG_1},$$

$$K_{21} = \frac{K_1 \breve{\alpha}_1 G_2}{(1+sG_1)^2}, \qquad \widetilde{K}_{22} = \frac{K_1 \breve{\alpha}_1 G_1}{1+sG_1} - \breve{\alpha}_2 G_3 - \hat{\delta}_1 - 2\eta_1 G_2 - \tilde{\gamma}_2,$$

$$\widetilde{K}_{23} = -\breve{\alpha}_2 G_2, \qquad \widetilde{K}_{32} = K_2 \breve{\alpha}_2 G_2, \qquad \widetilde{K}_{33} = K_2 \breve{\alpha}_2 G_2 - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3$$

Theorem 5.1. The equilibrium point E_0 exhibits local asymptotic stability. If $\widetilde{C} < \breve{\eta}_1$.

Proof: The Jacobian matrix J corresponding to the equilibrium point is given by E_0 has the eigenvalues

$$\lambda_1 = \tilde{C} - \breve{\eta}_1$$
$$\lambda_2 = -(\hat{\delta}_1 + \breve{\eta}_1)$$
$$\lambda_3 = -\hat{\delta}_2$$

It is clear that $\lambda_2 < 0$ and $\lambda_3 < 0$. Further, $\lambda_1 < 0$ if $C - \breve{\eta}_1$.

Hence, all eigenvalues have negative real parts under this condition.

Therefore, the equilibrium point E_0 is locally asymptotically stable.

Theorem 5.2. The equilibrium point E_1 exhibits local asymptotic stability if $\tilde{C}e < 2e(\tilde{C} - \breve{\eta}_1) - \breve{\eta}_1$.

Proof: The Jacobian matrix J corresponding to the equilibrium point E_1 is given by has the eigenvalues

$$\lambda_1 = \tilde{C}e - 2e(\tilde{C} - \breve{\eta}_1) - \breve{\eta}_1,$$

$$\lambda_2 = \frac{k_1\breve{\alpha}_1 e(c-\breve{\eta}_1)}{1+se(c-\breve{\eta}_1)} - (\hat{\delta}_1 + \breve{\eta}_2)$$
$$\lambda_3 = -\hat{\delta}_2$$

It is clear that $\lambda_2 = \frac{k_1\breve{\alpha}_1 e(c-\breve{\eta}_1)}{1+se(c-\breve{\eta}_1)} < (\hat{\delta}_1 + \breve{\eta}_2)$Further, $\lambda_1 = \tilde{C}e < 2e(\tilde{C} - \breve{\eta}_1) - \breve{\eta}_1$ if $\tilde{C} < \breve{\eta}_1$ and $\lambda_1 < 0$.

Hence, all eigenvalues have negative real parts under this condition. Therefore, the equilibrium point E_1 is locally asymptotically stable.

Theorem 5.3. The equilibrium point $E_2(G_1^*, G_2^*, 0)$ is locally asymptotically stable provided that

$$G_1^* < \frac{2\tilde{\gamma}_1 e}{\tilde{c}}, G_2^* > \frac{\hat{\delta}_2}{\breve{\alpha}_2\hat{k}_2}.$$

proof: The Jacobian matrix J corresponding to the equilibrium point $E(G_1, G_2, 0)$ is given by

$$J = \begin{bmatrix} \widetilde{K}_{11}^* & \widetilde{K}_{12}^* & 0 \\ \widetilde{K}_{21}^* & \widetilde{K}_{22}^* & \widetilde{K}_{23}^* \\ 0 & 0 & \widetilde{K}_{33}^* \end{bmatrix} \tag{5.3}$$

Here,
At the equilibrium point E_2, the Jacobian matrix has one eigenvalue given by

$$\lambda_1 = \widetilde{K}_{33}^*$$

and the remaining two eigenvalues are obtained from the quadratic equation

$$\lambda^2 - \breve{Q}_1\lambda + \breve{Q}_2 = 0$$

where $\breve{Q}_1$ and $\breve{Q}_2$ are functions of the system parameters.
For local asymptotic stability, all eigenvalues must have negative real parts. This holds if

$$\lambda_1 < 0,\ \breve{Q}_1 > 0,\ \breve{Q}_2 > 0.$$

The condition $\lambda_1 < 0$ gives $G_2^* > \frac{\hat{\delta}_2}{\breve{\alpha}_2\hat{k}_2}$.

Similarly, the conditions $\breve{Q}_1 > 0$ and $\breve{Q}_2 > 0$ are satisfied when

$$G_1^* < \frac{2\tilde{\gamma}_1 e}{\tilde{c}}$$

Thus, under these conditions, all eigenvalues possess negative real parts, implying that the equilibrium point E_2 is locally asymptotically stable.

Theorem 5.4. The coexistence equilibrium point E_3 exhibits local asymptotic stability if and only if the following conditions are verified.

$$\breve{A}_1^* > 0, \breve{A}_3^* > 0, \breve{A}_1^*\breve{A}_2^* - \breve{A}_3^* > 0$$

Proof: The system's Jacobian matrix (2.1) at E_3 is given by

$$J = \begin{bmatrix} \hat{K}_{11}^* & \hat{K}_{12}^* & 0 \\ \hat{K}_{21}^* & \hat{K}_{22}^* & \hat{K}_{23}^* \\ 0 & \hat{K}_{32}^* & \hat{K}_{33}^* \end{bmatrix} \tag{5.4}$$

Here,

$$\hat{K}_{11}^{*} = \tilde{C} - \frac{2\tilde{C}G_1^*}{e} - \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} - \breve{\eta}_1, \hat{K}_{12} = -\frac{\breve{\alpha}_1 G_1^*}{1+sG_1^*},$$

$$\hat{K}_{21} = \frac{\hat{K}_1\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2}, \quad \hat{K}_{22} = \frac{\hat{K}_1\breve{\alpha}_1 G_1^*}{1+sG_1^*} - \breve{\alpha}_2 G_3^* - \hat{\delta}_1 - 2\eta_1 G_2^* - \tilde{\gamma}_2,$$

$$\hat{K}_{23} = -\breve{\alpha}_2 G_2^*, \quad \hat{K}_{32} = \hat{K}_2\breve{\alpha}_2 G_2^*, \hat{K}_{33} = \hat{K}_2\breve{\alpha}_2 G_2^* - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*$$

At the equilibrium point E_3, the characteristic equation of the Jacobian matrix is expressed as

$$\lambda^3 + \breve{A}_1^*\lambda^2 - \breve{A}_2^*\lambda + \breve{A}_3^* = 0$$

Where,

$$\breve{A}_1^* = -\left(\hat{K}_{11}^* + \hat{K}_{22}^* + \hat{K}_{33}^*\right)$$

$$= \frac{2\tilde{C}G_1^*}{e} - \tilde{C} + \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} + \breve{\eta}_1 - \frac{\hat{K}_1\breve{\alpha}_1 G_1^*}{1+sG_1^*} + \breve{\alpha}_2 G_3^* + \hat{\delta}_1 + 2\eta_1 G_2^* + \tilde{\gamma}_2$$
$$-\hat{K}_2\breve{\alpha}_2 G_2^* + \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*$$

$$\breve{A}_2^* = \hat{K}_{33}^*\hat{K}_{22}^* - \hat{K}_{32}^*\hat{K}_{23}^* + \hat{K}_{11}^*\hat{K}_{33}^* + \hat{K}_{22}^*\hat{K}_{11}^* - \hat{K}_{12}^*\hat{K}_{21}^*$$

$$= \left(\hat{K}_2\breve{\alpha}_2 G_2^* - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*\right)\left(\frac{K_1\breve{\alpha}_1 G_1^*}{1+sG_1^*} - \breve{\alpha}_2 G_3^* - \hat{\delta}_1 - 2\eta_1 G_2^* - \tilde{\gamma}_2\right)$$
$$-\left(\left(\hat{K}_2\breve{\alpha}_2 G_2^*\right)\left(-\breve{\alpha}_2 G_2^*\right)\right)$$
$$+\left(\tilde{C} - \frac{2\tilde{C}G_1^*}{e} - \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} - \breve{\eta}_1\right)\left(\hat{K}_2\breve{\alpha}_2 G_2^* - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*\right)$$
$$+\left(\frac{\hat{K}_1\breve{\alpha}_1 G_1^*}{1+sG_1^*} - \breve{\alpha}_2 G_3^* - \hat{\delta}_1 - 2\eta_1 G_2^* - \tilde{\gamma}_2\right)$$
$$\times\left(\tilde{C} - \frac{2\tilde{C}G_1^*}{e} - \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} - \breve{\eta}_1\right)$$
$$-\left(-\frac{\breve{\alpha}_1 G_1^*}{1+sG_1^*}\right)\left(\frac{\hat{K}_1\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2}\right)$$

$$\breve{A}_3^* = \hat{K}_{11}^*\hat{K}_{23}^*\hat{K}_{32}^* + \hat{K}_{12}^*\hat{K}_{21}^*\hat{K}_{33}^* - \hat{K}_{11}^*\hat{K}_{33}^*\hat{K}_{22}^*$$
$$= \left(\tilde{C} - \frac{2\tilde{C}G_1^*}{e} - \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} - \breve{\eta}_1\right)\left(\hat{K}_2\breve{\alpha}_2 G_2^*\right)\left(-\breve{\alpha}_2 G_2^*\right)$$
$$+\left(-\frac{\breve{\alpha}_1 G_1^*}{1+sG_1^*}\right)\left(\frac{\hat{K}_1\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2}\right)\left(K_2\breve{\alpha}_2 G_2^* - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*\right)$$
$$-\left(\hat{K}_2\breve{\alpha}_2 G_2^* - \hat{\delta}_2 - 2\tilde{\gamma}_2 G_3^*\right)$$
$$\times\left(\tilde{C} - \frac{2\tilde{C}G_1^*}{e} - \frac{\breve{\alpha}_1 G_2^*}{(1+sG_1^*)^2} - \breve{\eta}_1\right)$$
$$\times\left(\frac{\hat{K}_1\breve{\alpha}_1 G_1^*}{1+sG_1^*} - \breve{\alpha}_2 G_3^* - \hat{\delta}_1 - 2\eta_1 G_2^* - \tilde{\gamma}_2\right)$$

Based on the Routh-Hurwitz criterion, the coexistence equilibrium point E_3 exhibits local asymptotic stability provided that $\breve{A}_1^* > 0, \breve{A}_3^* > 0$, and $\breve{A}_1^*\breve{A}_2^* - \breve{A}_3^* > 0$ are satisfied.

Theorem 5.5. Under the conditions listed below, the equilibrium point $E_3(G_1^*, G_2^*, G_3^*)$, at which all species coexist in system (2.1), is globally asymptotically stable.

$$Q_1 = \frac{1}{\hat{k}_1}, Q_2 = \frac{Q_1}{\hat{k}_2}$$

proof: The proof can be obtained by applying the Lyapunov stability theorem. Let us now consider a positive definite function R is characterized as

$$\check{V} = (G_1 - G_1^*) - G_1^* \text{In}\left(\frac{G_1}{G_1^*}\right) + Q_1(G_2 - G_2^*) - G_2^* \text{In}\left(\frac{G_2}{G_2^*}\right) + Q_2(G_3 - G_3^*) - G_3^* \text{In}\left(\frac{G_3}{G_3^*}\right)$$

Taking the time derivative of V along the trajectories of the system (2.1), we obtain

$$\frac{dV}{dt} = \left(\frac{G_1 - G_1^*}{G_1}\right)\frac{dG_1}{dt} + Q_1\left(\frac{G_2 - G_2^*}{G_2}\right)\frac{dG_2}{dt} + Q_2\left(\frac{G_3 - G_3^*}{G_3}\right)\frac{dG_3}{dt}$$

Now by substituting the model equation of system (2.1), we get

$$\frac{d\check{V}}{dt} = \left(\frac{G_1 - G_1^*}{G_1}\right)\left[\tilde{C}G_1\left(1 - \frac{G_1}{e}\right) - \frac{\breve{\alpha}_1 G_1 G_2}{(1 + G_1\bar{S}_1)} - \breve{\eta}_1 G_1\right]$$
$$+\left(\frac{G_2 - G_1^*}{G_2}\right)\left[\frac{\widehat{K}_1\breve{\alpha}_1 G_1 G_2}{(1 + G_1\bar{S}_1)} - \hat{\delta}_1 G_2 - \breve{\alpha}_2 G_2 G_3 - \tilde{\gamma}_1 G_2^2 - \breve{\eta}_2 G_2\right]$$
$$+\left(\frac{G_3 - G_3^*}{G_3}\right)\left[\widehat{K}_2\breve{\alpha}_2 G_2 G_3 - \hat{\delta}_2 G_3 - \tilde{\gamma}_2 G_3^2\right]$$

$$\frac{d\check{V}}{dt} = -\frac{\tilde{C}}{e}(G_1 - G_1^*)^2 + \left[\frac{(Q_1\hat{k}_1 - 1)\breve{\alpha}_1}{(1 + sG_1)(1 + sG_1^*)}\right](G_1 - G_1^*)(G_2 - G_2^*)$$
$$-\breve{\eta}_1(G_1 - G_1^*)$$
$$-\left[\frac{\breve{\alpha}_1 s(G_2 G_1^* - G_2^* G_1)}{(1 + sG_1)(1 + sG_1^*)}\right](G_1 - G_1^*)(G_2 - G_2^*)$$
$$-\tilde{\gamma}_1(G_2 - G_2^*)^2 + \breve{\alpha}_2(Q_2\hat{k}_2 - Q_1)(G_2 - G_2^*)$$
$$(G_3 - G_3^*) - \tilde{\gamma}_2(G_3 - G_3^*)^2$$

By choosing the constants $Q_1 = \frac{1}{\hat{k}_1}$ and $Q_2 = \frac{Q_1}{\hat{k}_2}$, the cross terms vanish or become non-positive. Hence, we obtain $\frac{d\check{V}}{dt} \le 0$.

Further, $\frac{d\check{V}}{dt} = 0$ holds only at the equilibrium point E_3. Therefore, using a Lyapunov function, Global asymptotic stability is attained at the equilibrium point E_3.[17]

6. Numerical Simulations

For the numerical solution, consider system (2.1) $\tilde{C} = 1, e = 2, S_1 = 1.5, \breve{\alpha}_1 = 2.5, \breve{\alpha}_2 = 0.8, \widehat{K}_1 = 0.6, \widehat{K}_2 = 0.5, \hat{\delta}_1 = 0.4 \tilde{\gamma}_1 = 0.05, \tilde{\gamma}_2 = 0.5, \breve{\eta}_2 = 0.02, \breve{\eta}_1 = 0.01 \hat{\delta}_2 = 0.03, G_1 = 0.4552, G_2 = 1.8260, G_3 = 1.4008$. The numerical values of Thorem:5.4, $\breve{A}_1^* = 2.6371$ and $\breve{A}_3^* = 3.6954$ are obtained, and the condition $\breve{A}_1^*\breve{A}_2^* - \breve{A}_3^* > 1.6394$ is satisfied. To examine the system dynamics (2.1), numerical simulations are carried out using suitable parameter values. The solutions are obtained using matlab.$(G_1^*, G_2^*, G_3^*) = (1.2729, 0.4117, 0.2694)$. It is observed that all state variables converge asymptotically to their equilibrium values, confirming that E_3 is locally asymptotically stable. The corresponding phase diagram is presented in Figure 2, which further confirms the stability of the system as Figure 1 depicts the time series of the system corresponding to the coexistence equilibrium point all trajectories approach the equilibrium point.To examine the influence of parameter variations, we consider $\hat{\delta}_1 = 0.14$ and $\tilde{\gamma}_1 = 0$. The resulting dynamics are shown in Figure 3, where the system again approaches the equilibrium point $(G_1^*, G_2^*, G_3^*) = (1.0434, 0.3471, 0.2376)$, indicating stable coexistence. Further, by setting $\tilde{\gamma}_2 = 0$ and $\hat{\delta}_1 = 0.14$, the system behavior changes significantly, as shown in Figure 4. In this case, the predator population G_2 tends to zero, while G_1 and G_3 approach the equilibrium (1.72004,0,0.2893), indicating partial extinction of the secondary species. Finally, when $\tilde{\gamma}_2$ is increased to 0.02 while keeping $\hat{\delta}_1 = 0.14$, the system stabilizes at $(G_1^*, G_2^*, G_3^*) =$

(1.72014,0,0.1030), as illustrated in Figure 5. This highlights the crucial influence of intra-specific competition in regulating the dynamics of the top predator population.

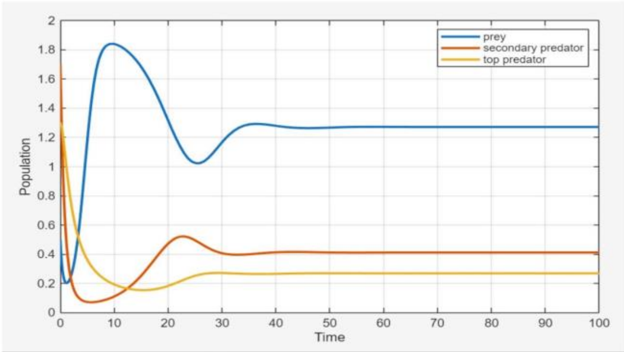

Figure 1: $E_3 = (G_1^* = 1.2729, G_2^* = 0.4117, G_3^* = 0.2694)$ exhibits local asypotic stablility

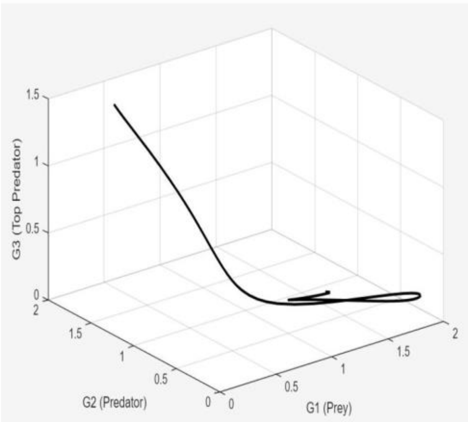

Figure 2: Phase diagram

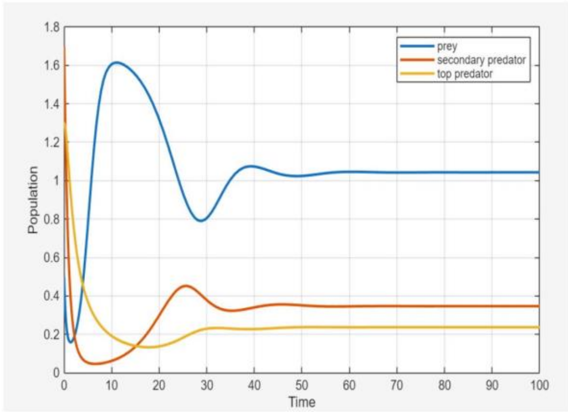

Figure 3: $\hat{\delta}_1 = 0.14, \tilde{\gamma}_1 = 0, (G_1^* = 1.0434, G_2^* = 0.3471, G_3^* = 0.2376)$

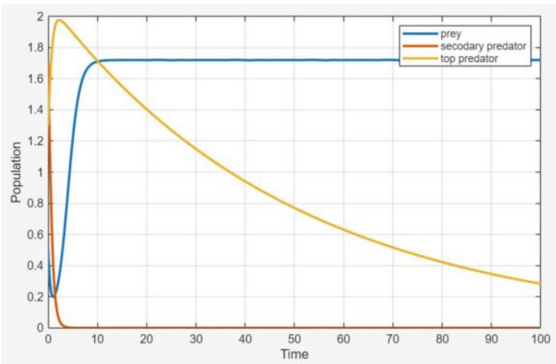

Figure 4 : $\tilde{\gamma}_2 = 0, \tilde{\gamma}_2 = 0, \hat{\delta}_1 = 0.14, (G_1^* = 1.72004, G_2^* = 0, G_3^* = 0.2893)$

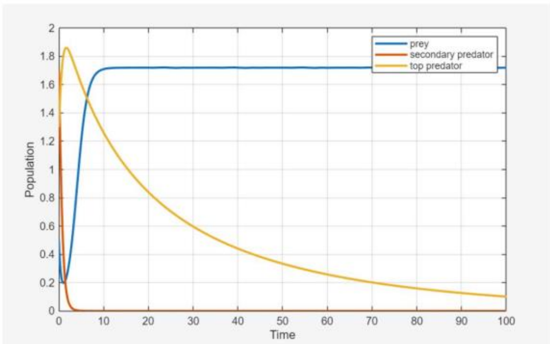

Figure 5 : $\tilde{\gamma}_2 = 0, \tilde{\gamma}_2 = 0.02, \hat{\delta}_1 = 0.14, (G_1^* = 1.72014, G_2^* = 0, G_3^* 0.1030)$

7. Conclusion

This paper focuses on a dynamic model of a three-species ecosystem consisting of prey, secondary predator, and top predator. with nonlinear functional response and intra-specific competition has been investigated. The equilibrium points have been examined for their existence and stability through analytical methods. The results indicate that the coexistence equilibrium point is locally asymptotically stable under certain parametric conditions. The influence of key parameters, such as predation rates, mortality rates, and intra-specific competition coefficients, serves as an important factor in determining the system's long-term evolution.

Simulations are implemented to demonstrate consistency with the analytical results. The time series and phase portraits confirm that all populations converge to their steady-state values when stability conditions are satisfied. Further, variations in parameters demonstrate significant changes in system behavior, including species extinction and reduced population levels.

In particular, the effect of intra-specific competition on the system has been analyzed. It is observed that the absence of intra-specific competition leads to instability, resulting in uncontrolled growth, oscillations, or extinction of species.

However, when intra-specific competition parameters are introduced, the system dynamics becomes stable and all species tend to coexist at equilibrium. The competition within the same species effectively regulates population growth and prevents ecological imbalance.

Hence, intra-specific competition has a vital influence on both the stability and long-term persistence of the system.

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