A Non - Markovian Queue with Two Types of Phases Services, with an Optional Service and Balking

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Abstract: A single server non -Markovian queue with two types of essential services of phase type and with an optional service has been considered. In which the customer May balk. The model in analyzed in study state. Some numerical results are presented.

Keywords: *Non-Markovian queue-Phase services-Essential service-Optional service- Bernoulli process-Balking.*

Mathematics Subject Classification Number: 90B22,60K25,60K30.

1 Introduction

In a non Markovian queuing system the inter arrival time or the service time or the both inter arrival time and service time follows general distribution. The literature on non-Markovian queue is rich. Many author working on such system. The service mechanism, based on real life situation, can have different categories depends on the customers need. For example, the service type may be fast or slow or detailed service. This types of queues are called system with multitype of services. Another modification, based on the satisfaction level of the customer, the customer may receive essential service and/or optional second service. The above mentioned service mechanisms are common in the (i) Hospital system: Emergency treatment and regular treatment (ii) Computer service: Quick requests and Heavy Processing jobs (iii) Manufacturing: Normal processing and rework/inspection. Also, the service may be divided into different phases and transitions occur between phases until the service completion.

The customer behavior may have impact on a queuing system. One of the customer behaviors is balking. On arrival, the customer may or may not join the queue based on the system condition, called balking behavior.

Several authors working on queue with different types of services and different phase types of service. Madan and Ayman Baklizi (2002) have studied M/G/1 queue with additional second stage service and optional re-service. Bademchi zadeh (2009) and Jain and Upadhyaya (2010) discussed about second optional service. Bayoush et al (2013) studied a model with two stages of heterogeneous services.

Madhan (2000) studied a queue with first essentially service which is generally distributed and second optional service which is exponentially distributed. Medhi (2002) studied a queue with both essential as well as optional service and generally distributed.

In a queuing system,customers may decide not to join the queue when they see that it is too long or the waiting time is too high, this behavior is called balking behavior. Haight (1957) was first to developed a queue where a customer can balk and after that in (1959). Kulla (2013) has presented a queues with balking and reneging and Medhi (2020) has presented queues with balking and reneging.

The model defined in this article is a single server non-Markovian queue with two types of essential services, each service has multistage phases. After completing second type service, the server goes for optional service. In addition, the customer may balk. This model is analyzed in steady state using supplementary variable technique.

The following real situation can be perfectly fit to the model discussed in this article: In a vehicle centre, the vehicles are serviced using the phases namely inspection,repair after which it has optional polishing. If the service centre is crowded, the vehicle may leave without servicing. In a IT support centre, the phases namely issue logging and troubleshooting are the essential part of service and optional premium support is the optional support. If waiting is long the users may abandon. In a Bank, the service is, Bank loan processing has the phases verification and approval are essential part and optional insurance is the optional part. The customer may balk if the queue is long.

The model definition is given in section 2, the analysis is given in section 3, and some performance measures are given in section 4. The model is numerically analyzed in section 5 and a conclusion is given in section 6.

The model defined in this paper has the following characters:

There is a single server providing two types of essential services and an optional service. Each types of essential service has multiple phases where as the optional service is single phase.The server may Balk.

2 The Model

The arrival follows Poisson with rate λ and a single server provides two types of services (type 1 and type 2). Each type of service has multi-stages of heterogeneous services one after another. The entering customer selects type 1 service with probability p or type 2 service with probability 1-p. After completion of type 1 service, the customer leaves the system, where as after completion of type 2 service the customer leaves the system with probability 1-r or choose an optional service with probability r. After completion of optional service, the customer leaves the system. An arriving customer may balk with probability q, if service is not immediate or join the queue with probability 1-q

The schematic representation of the model is given in the following figure.

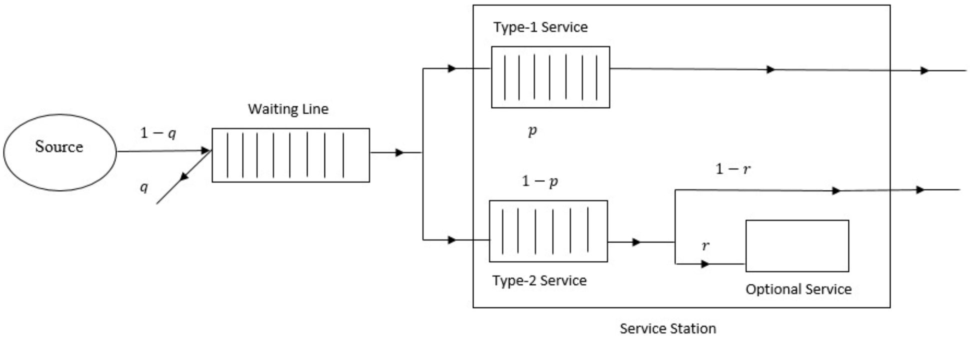

Figure 2.1:Schematic representation of the model

The service time distributions are general, the distribution functions are $B_{1,j}(x)$ for type 1 and j^{th} phase of service (j = 1, 2, …, N) and $B_{2,j}(x)$ for type 2 service and j^{th} phase of service (j = 1,2 …., M) and $B_2(x)$ for optional service. The hazard rate functions are

$\mu_{1,j}(x) = \frac{b_{1,j}(x)}{1-B_{1,j}(x)}$; j=1,2,…..,N

$\mu_{2,j}(x) = \frac{b_{2,j}(x)}{1-B_{2,j}(x)}$; j=1,2,…..,M Where $b_{1,j}(x) = dB_{1,j}(x)$; i=1,2

$\mu_2(x) = \frac{b_2(x)}{1-B_2(x)}$,Where $b_2(x) = dB_2(x)$

Let K (t) be the number of customers in the queue at time t and $\xi(t)$ be the server state at time t, where $\xi(t)$= 0, the server is idle, $\xi(t)$ = (1, j), the server is j^{th} phase type 1 service,$\xi(t)$ = (2,j), the server is j^{th} phase type 2 service and $\xi(t)$ = 2, the server is optional type of service.

Now we introduce the elapsed time $Y_{ij}(t)$ as supplementary variable i=0,1,2. The two dimensional process $\{(K(t), Y_{ij}(t)): t \geq 0\}$ is a two dimensional Markov process. The probabilities are defined as:

$Q(t) = Pr\{ K(t) = 0, \xi(t) = 0 \}$

$P_n^{(1,j)}(x;t)\, dt = Pr \{ K(t) = n ; t \leq Y_{1j}(t) \leq t + dt \}$; j = 1, 2, …, N.

$P_n^{(2,j)}(x;t)\, dt = Pr \{ K(t) = n ; t \leq Y_{2j}(t) \leq t + dt \}$; j = 1, 2, …, M.

$P_n^{(2)}(x;t)\, dt = Pr \{ K(t) = n ; t \leq Y_{3j}(t) = Y_2(t) \leq t + dt \}$

In Steady state,

$$Q = \lim_{t\to\infty} Q(t), P_n^{(1,j)}(x) = \lim_{t\to\infty} P_n^{(1,j)}(x;t);\ i = 1,2, P_n^{(2)}(x) = \lim_{t\to\infty} P_n^{(2)}(x;t)$$

The Corresponding probability generating functions are

$$P^{(1,j)}(x,z) = \sum_{n=0}^{\infty} P_n^{(1,j)}(x) z^n;\quad j = 1,2,\ldots..N;\quad |z| \leq 1$$

$$P^{(2,j)}(x,z) = \sum_{n=0}^{\infty} P_n^{(2,j)}(x) z^n;\quad j = 1,2,\ldots..M;$$

$$P^{(2)}(x,z) = \sum_{n=0}^{\infty} P_n^{(2)}(x) z^n$$

3 The Analysis

The supplementary variable technique adopted by Cox (1955) has been used to derive the following differential - difference equations:

$\frac{d}{dx}P_0^{(1,j)}(x) + \left((1-q)\,\lambda + \mu_{1,j}(x)\right)P_0^{(1,j)}(x) = 0; j = 1,2,\ldots\ldots,N$

(3.1)

$\frac{d}{dx}P_n^{(1,j)}(x) + \left((1-q)\lambda + \mu_{1,j}(x)\right)P_n^{(1,j)}(x) = (1-q)\lambda\, P_{n-1}^{(1,j)}(x);\ n \geq 1 \quad j = 1,2,\ldots\ldots,N$ (3.2)

$\frac{d}{dx}P_0^{(2,j)}(x) + \left((1-q)\lambda + \mu_{2,j}(x)\right)P_0^{(2,j)}(x) = 0; j = 1,2,\ldots\ldots,M$

(3.3)

$\frac{d}{dx}P_n^{(2,j)}(x) + \left((1-q)\,\lambda + \mu_{2,j}(x)\right)P_n^{(2,j)}(x) = (1-q)\lambda\, P_{n-1}^{(2,j)}(x);\ n \geq 1 \quad j = 1,2,\ldots\ldots,M$ (3.4)

$\frac{d}{dx}P_0^{(2)}(x) + ((1-q)\lambda + \mu_2(x))P_0^{(2)}(x) = 0$

(3.5)

$\frac{d}{dx}P_n^{(2)}(x) + ((1-q)\,\lambda + \mu_2(x))P_n^{(2)}(x) = (1-q)\lambda P_{n-1}^{(2)}(x);\ n \geq 1$

(3.6)

$\lambda\,(1-q)Q = \int_0^\infty P_0^{(1,N)}(x)\mu_{1,N}(x)dx + (1-r)\int_0^\infty P_0^{(2,M)}(x)\mu_{2,M}(x)dx + \int_0^\infty P_0^{(2)}(x)\mu_2(x)$ (3.7)

The boundary conditions are

$P_0^{(1,1)}(0) = \lambda\,(1-q)pQ + p\int_0^\infty P_1^{(1,N)}(x)\mu_{1,N}(x)dx + (1-r)p\int_0^\infty P_1^{(2,M)}(x)\mu_{2,M}(x)dx +$
$p\int_0^\infty P_1^{(2)}(x)\mu_2(x)dx$ (3.8)

$P_n^{(1,1)}(0) = p\int_0^\infty P_{n+1}^{(1,N)}(x)\mu_{1,N}(x)dx + (1-r)p\int_0^\infty P_{n+1}^{(2,M)}(x)\mu_{2,M}(x)dx + p\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx; n \geq 1$

(3.9)

$P_n^{(1,j)}(0) = \int_0^\infty P_n^{(1,j-1)}(x)\mu_{1,1-j}(x)dx\,;\quad n \geq 1,\quad j = 2,\ldots\ldots,N$

(3.10)

$P_0^{(2,1)}(0) = \lambda(1-q)(1-p)\,Q + (1-p)\int_0^\infty P_1^{(1,N)}(x)\mu_{1,N}(x)dx + (1-r)(1-p)\int_0^\infty P_1^{(2,M)}(x)\mu_{2,M}(x)dx +$
$(1-p)\int_0^\infty P_1^{(2)}(x)\mu_2(x)dx$

(3.11)

$P_n^{(2,1)}(0) = (1-p)\int_0^\infty P_{n+1}^{(1,N)}(x)\mu_{1,N}(x)dx + (1-r)(1-p)\int_0^\infty P_{n+1}^{(2,M)}(x)\mu_{2,M}(x)dx + (1-$
$p)\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx;\quad n \geq 1$

(3.12)

$P_n^{(2,j)}(0) = \int_0^\infty P_n^{(2,j-1)}(x)\mu_{2,j-1}(x)dx\,; \quad n \geq 1,\ j = 2,\ldots\ldots,M$

(3.13)

$P_n^{(2)}(0) = r\int_0^\infty P_n^{(2,M)}(x)\mu_{2,M}(x)dx\,; \quad n \geq 0$

(3.14)

The normalization conditions is

$Q + \sum_{n=0}^{\infty}\sum_{j=1}^{N}\int_0^\infty P^{(1,j)}(x)\,dx + \sum_{n=0}^{\infty}\sum_{j=1}^{M}\int_0^\infty P^{(2,j)}(x)\,dx + \sum_{n=0}^{\infty}\int_0^\infty P^{(2)}(x)\,dx = 1$

(3.15)

Multiplying equations (3.2), (3.4) and (3.6) by z^n, summing from n = 1 to ∞ and then adding (3.1), (3.3) and (3.5),

we get

$\frac{\frac{d}{dx}P^{(1,j)}(x,z)}{P^{(1,j)}(x,z)} = [-s - \mu_{1,j}(x)]; \qquad j = 1,2,\ldots\ldots,N)$

(3.16a)

$\frac{\frac{d}{dx}P^{(2,j)}(x,z)}{P^{(2,j)}(x,z)} = [-s - \mu_{2,j}(x)]; \qquad j = 1,2,\ldots\ldots,M$

(3.16b)

$\frac{\frac{d}{dx}P^{(2)}(x,z)}{P^{(2)}(x,z)} = [-s - \mu_2(x)]; Where \quad s = \lambda\,(1-q)(1-z)$

(3.16c)

Integration of the above equations leads to,

$P^{(1,j)}(x,z) = A_{1,j}\left(1 - B_{1,j}(x)\right)e^{-sx}; \quad j = 1,2.\ldots.,N$

(3.17a)

$P^{(2,j)}(x,z) = A_{2,j}\left(1 - B_{2,j}(x)\right)e^{-sx}; \quad j = 1,2.\ldots.,M$

(3.17b)

$P^{(2)}(x,z) = A_2\big(1 - B_2(x)\big)e^{-sx}$

(3.17c)

Taking x=0 in equations (3.17a), (3.17b) and (3.17c), constants A are obtained as

$A_{1,j} = P^{(1,j)}(0,z); \quad j = 1,2.\ldots.,N$

(3.18a)

$A_{2,j} = P^{(2,j)}(0,z); \quad j = 1,2.\ldots.,M$

(3.18b)

$A_2 = P^{(2)}(0,z)$
(3.18c)

Using equations (3.18a), (3.18b) and (3.18c) in (3.17a), (3.17b) and (3.17c),we get

$P^{(1,j)}(x,z) = P^{(1,j)}(0,z)\left(1 - B_{1,j}(x)\right)e^{-sx}; \quad j = 1,2.....,N$
(3.19a)

$P^{(2,j)}(x,z) = P^{(2,j)}(0,z)\left(1 - B_{2,j}(x)\right)e^{-sx}; \quad j = 1,2.....,M$
(3.19b)

$P^{(2)}(x,z) = P^{(2)}(0,z)\left(1 - B_2(x)\right)e^{-sx}$
(3.19c)

Multiplying equations (3.9) by z^n, summing from n=1 to ∞ adding (3.8), using (3.7), (3.19a), (3.19b) and (3.19c),

we get

$zP^{(1,1)}(0, z) = \lambda(1-q)(z-1)Q + pB_{1,N}^*(s)P^{(1,N)}(0,z) + p(1-r)B_{2,M}^*(s)P^{(2,M)}(0,z) + pB_2^*(s)P^{(2)}p(0,z)$(3.20)

Similarly, equations (3.10) to (3.14) and using (3.7),(3.19a),(3.19b) and (3.19c),we get

$P^{(1,j)}(0, z) = P^{(1,j-1)}(0,z)B_{1,j-1}^*(s); \quad j = 2,.......,N$
(3.21)

$zP^{(2,1)}(0,z) = \lambda(1-q)(1-p)(z-1)Q + (1-p)B_{1,N}^*(s)P^{(1,N)}(0,z) + (1-p)(1-r)B_{2,M}^*(s)P^{(2,M)}(0,z) +$

$(1-p)B_2^*(s)P^{(2)}p(0,z)$ (3.22)

$P^{(2,j)}(0, z) = P^{(2,j-1)}(0,z)B_{2,j-1}^*(s); \quad j = 2,.......,M$
(3.23)

$P^{(2)}(0,z) = rP^{(2,M)}(0,z)B_{2,M}^*(s)$ (3.24)

From (3.21) and (3.23), we get

$P^{(1,N)}(0,z) = B_{1,N-1}^*(s)B_{1,N-2}^*(s).....B_{1,1}^*(s)P^{(1,1)}(0,z)$ (3.25)

$P^{(2,M)}(0,z) = B_{2,M-1}^*(s)B_{2,M-2}^*(s).....B_{2,1}^*(s)P^{(2,1)}(0,z)$
(3.26)

Applying(3.26) in (3.24), we get

$P^{(2)}(0,z) = rB_{2,M}^*(s)B_{2,M-1}^*(s)B_{2,M-2}^*(s).....B_{2,1}^*(s)P^{(2,1)}(0,z)$(3.27)

Using equation (3.25),(3.26),(3.27) in (3.20) and (3.22), we get

$\left[z - pB_{1,N}^*(s)B_{1,N-1}^*(s)B_{1,N-2}^*(s)B_{1,N}^*(s).....B_{1,1}^*(s)\right]P^{(1,1)}(0,z) = \lambda(1-q)p(z-1)Q + p\left(1 - r + rB_2^*(s)\right)pB_{1,M}^*(s)B_{2,M-1}^*(s)B_{2,M-2}^*(s).....B_{2,1}^*(s)P^{(2,1)}(0,z)$ (3.28)

$$\left[z - (1-p)\left(1 - r + rB_2^*(s)\right)B_{2,M}^*(s)B_{2,M-1}^*(s)B_{2,M-2}^*(s).....B_{2,1}^*(s)\right]P^{(2,1)}(0,z) = \lambda(1-q)(1-p)(z-1)Q +$$

$(1-p)B_{1,N}^*(s)B_{1,N-1}^*(s)B_{1,N-2}^*(s).....B_{1,1}^*(s)P^{(1,1)}(0,z)$

(3.29)

Solving equations (3.28) and (3.29),we get

$P^{(1,1)}(0,z) = \frac{\lambda(1-q)p(z-1)Q}{D(z)}$

(3.30)

$P^{(2,1)}(0,z) = \frac{\lambda(1-q)(1-p)(z-1)Q}{D(z)}$

(3.31)

Where,$D(z) = z - pB_{1,N}^*(s)B_{1,N-1}^*(s)B_{1,N-2}^*(s).....B_{1,1}^*(s) - (1-p)\left(1 - r + rB_2^*(s)\right)B_{2,M}^*(s)B_{2,M-1}^*(s)B_{2,M-2}^*(s).....B_{2,1}^*(s)$

Substituting equations (3.30), and (3.31) in (3.21), we get

$P^{(1,2)}(0,z) = \frac{B_{1,1}^*(s)\lambda(1-q)p(z-1)Q}{D(z)}$.

$(0,z) = \frac{B_{1,N-1}^*(s)B_{1,N-2}^*(s).....B_{1,2}^*(s)B_{1,1}^*(s)\lambda(1-q)p(z-1)Q}{D(z)}$ (3.32)

In general

$P^{(1,j)}(0,z) = \frac{B_{1,j-1}^*(s)B_{1,j-2}^*(s).....B_{1,1}^*(s)\lambda(1-q)p(z-1)Q}{D(z)}; \quad j = 2,.....,N$

(3.33)

Substituting equations (3.31) in (3.23), we get

$P^{(2,2)}(0,z) = \frac{B_{2,1}^*(s)\lambda(1-q)(1-p)(z-1)Q}{D(z)}$

$P^{(2,M)}(0,z) = \frac{B_{2,M-1}^*(s).....B_{2,1}^*(s)\lambda(1-q)(1-p)(z-1)Q}{D(z)}$ (3.34)

In general

$P^{(2,j)}(0,z) = \frac{B_{2,j-1}^*(s)B_{2,j-2}^*(s).....B_{2,1}^*(s)\lambda(1-q)(1-p)(z-1)Q}{D(z)}; \quad j = 2,.....,M$ (3.35)

Substituting the value of $P^{(2,M)}(0,z)$ in (3.24),we get

$P^{(2)}(0,z) = \frac{rB_{2,M}^*(s)B_{2,M-1}^*(s).....B_{2,1}^*(s)\lambda(1-q)(1-p)(z-1)Q}{D(z)}$

(3.36)

Integration of equation (3.19) by parts with respect to x and using equation (3.32) to(3.36),we get

$P^{(1,j)}(z) = \int_0^\infty P^{(1,j)}(x,z)dx$

$P^{(1,1)}(z) = \frac{pQ\left(B_{1,1}^*(s)-1\right)}{D(z)}$

(3.37)

$P^{(1,j)}(z) = \frac{pQB_{1,j-1}^*(s)B_{1,j-2}^*(s).....B_{1,1}^*(s)\left(B_{1,j}^*(s)-1\right)}{D(z)}; \quad j = 2, \ldots.., N$

(3.38)

$P^{(2,j)}(z) = \int_0^\infty P^{(2,j)}(x,z)dx$

$P^{(2,1)}(z) = \frac{(1-p)Q\left(B_{2,1}^*(s)-1\right)}{D(z)}$

(3.39)

$P^{(2,j)}(z) = \frac{(1-p)QB_{2,j-1}^*(s)B_{2,j-2}^*(s).....B_{2,1}^*(s)\left(B_{2,j}^*(s)-1\right)}{D(z)}; \quad j = 2, \ldots.., M$

(3.40)

$P^{(2)}(z) = \int_0^\infty P^{(2)}(x,z)dx$

$P^{(2)}(z) = \frac{r(1-p)QB_{2,M}^*(s)B_{2,M-1}^*(s).....B_{2,1}^*(s)(B_2^*(s)-1)}{D(z)}$

(3.41)

The idle probability Q is obtained using the normalization condition

$Q + \sum_{j=1}^{N} P^{(1,j)} + \sum_{j=1}^{M} P^{(2,j)} + P^{(2)} = 1$

$Q = 1 - \rho$

(3.42)

Where, $\rho = \lambda(1-q)p\sum_{j=1}^{N} E\left(B_{1,j}\right) + \lambda(1-q)(1-p)\sum_{j=1}^{M} E\left(B_{2,j}\right) + \lambda(1-q)r(1-p)E(B_2)$

The utilization factor $\rho < 1$ is the stability condition under which the steady state solution exists.

Equations (3.37) to (3.41) together with equation (3.42) are respectively, the probability generating functions of the number of customers in the queue when the server is serving type 1 phase 1 service, serving type 1 phase *j* service, and serving type 2 phase 1 service, type 2 phase *j* and an optional service.

The probability generating functions for the number of customers in the queue irrespective of server state is

$$U(z) = Q + \sum_{j=1}^{N} P^{(1,j)}(z) + \sum_{j=1}^{M} P^{(2,j)}(z) + P^{(2)}(z)$$

$$U(z) = \frac{(z-1)(1-\rho)}{D(z)}$$

4The Performance Measures

Using straight forward calculations, the following performance measures have been obtained:

1. The mean number of customers in the queue is

$$L_q = \lim_{z\to 1}\frac{d}{dz}U(z)$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{(z-1)Q}{D(z)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{D(z)-(z-1)D'(z)}{(D(z))^2}\right]Q$$

(4.1)

Since this limit gives $\frac{0}{0}$ form, so applying L' Hospital rule twice, we get

$$L_q = \lim_{z\to 1}\frac{-D''(1)}{2\left(D'(1)\right)^2}Q$$

$$L_q = \lim_{z\to 1}\frac{(\lambda(1-q))^2 C_1}{2(1-\rho)}$$

Where,

$$C_1 = (1-p)\left[\sum_{j=1}^{M}\left(\sum_{i=1,i\neq j}^{M}E\left(B_{2,i}\right)E\left(B_{2,j}\right)+E\left(B_{2,j}^2\right)\right)+rE(B_2^2)+2rE(B_2)\sum_{j=1}^{M}E\left(B_{2,j}\right)\right]$$
$$+\ p\left[\sum_{j=1}^{N}\left(\sum_{i=1,i\neq j}^{N}E(B_{1,i})E(B_{1,j})+E\left(B_{1,j}^2\right)\right)\right]$$

2. The Variance of the number of outcomes on the queue is

$$V_{L_q} = \lim_{z\to 1}\frac{d^2}{dz^2}U(z)+\lim_{z\to 1}\frac{d}{dz}U(z)-\left(\lim_{z\to 1}\frac{d}{dz}U(z)\right)^2$$

Differentiating equation (4.1), with respect to *z*, the limit value leads to $\frac{0}{0}$ form, so applying L' Hospital rule four times, we get

$$\lim_{z\to 1}\frac{d^2}{dz^2}U(z) = \frac{\left[-2D'(1)D'''(1)+3\left(D''(1)\right)^2\right]Q}{6\left(D'(1)\right)^3}$$

Hence$U(1) = \frac{((1-q)\lambda)^2[2(1-\rho)C_2+3\lambda(1-q)C_1^2]}{6\left((1-\rho)\right)^2}$

$$V_{L_q} = \frac{((1-q)\lambda)^2[2(1-\rho)(2\lambda(1-q)C_2+3C_1)+3((1-q)\lambda)^2C_1^2]}{12\left((1-\rho)\right)^2} \quad (4.3)$$

Where,$C_2 = (1-p)[\sum_{j=1}^{M}E\left(B_{2,j}^3\right)+rE(B_2^3)+3rE(B_2)\sum_{j=1}^{M}\sum_{i=1,i\neq j}^{M}E\left(B_{2,i}\right)E\left(B_{2,j}\right)+$
$3rE(B_2)\sum_{j=1}^{M}E\left(B_{2,j}^2\right)+3rE(B_2^2)\sum_{j=1}^{M}E\left(B_{2,j}\right)+\sum_{j=1}^{M}\sum_{i=1,i\neq j}^{M}\sum_{k=1,k\neq i,k\neq j}^{M}E\left(\boldsymbol{B}_{2,k}\right)E\left(B_{2,i}\right)E(2,j)+$
$\sum_{j=1}^{M}\sum_{i=1,i\neq j}^{M}E\left(B_{2,i}^2\right)E\left(B_{2,j}\right)+2\sum_{j=1}^{M}\sum_{i=1}^{M}E\left(B_{2,j}^2\right)E\left(B_{2,i}\right)]+p[\sum_{j=1}^{N}E\left(B_{1,j}^3\right)+$

$\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}E\left(B_{1,i}^{2}\right)E\left(B_{1,j}\right)+2\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}E\left(B_{1,i}\right)E\left(B_{1,j}^{2}\right)+$
$\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}\sum_{k=1,k\neq i,k\neq j}^{N}E\left(B_{1,k}\right)E\left(B_{1,i}\right)E\left(B_{1,j}\right)\Big]$

3.The mean waiting time in the queue is

$W_q=\frac{L_q}{\lambda}$ (4.4)

4.The varience of the waiting time in the queue is

$\boldsymbol{V_{W_q}}=\frac{V_{L_q}}{\lambda^2}$(4.5)

5. The mean number of customers in the queue when the server provides type 1serviceis

$$L_{q1}=\lim_{z\to 1}\frac{d}{dz}\left[P^{(1,1)}(z)+\sum_{j=2}^{N}P^{(1,j)}(z)\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{pN_1(z)Q}{D(z)}\right]$$

Where $N_1(z)=B_{1,N}^{*}(s)B_{1,N-1}^{*}(s)B_{1,N-2}^{*}(s)\ldots..B_{1,1}^{*}(s)-1$

$$L_{q1}=\frac{pQ[D'(1)N_1''(1)-N_1'(1)D''(1)]}{2\left(D'(1)\right)^2}$$

$L_{q1}=\lim_{z\to 1}\frac{((1-q)\lambda)^2p}{2(1-\rho)}\Big[\left[\sum_{j=1}^{N}E\left(B_{1,j}^{2}\right)+\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}E\left(B_{1,i}\right)E\left(B_{1,j}\right)\right]-\lambda(1-q)(1-p)\left[\sum_{j=1}^{M}E\left(B_{2,j}\right)+\right.$
$\left.rE(B_2)\right]\left[\sum_{j=1}^{N}E\left(B_{1,j}^{2}\right)+\sum_{i=1}^{N}\sum_{i=1,i\neq j}^{N}E\left(B_{1,i}\right)E\left(B_{1,j}\right)\right]+\lambda(1-q)(1-p)\left[\sum_{j=1}^{N}E\left(B_{1,j}\right)\right]\left[\sum_{j=1}^{M}E\left(B_{2,j}^{2}\right)+\right.$
$\left.rE(B_2^2)+\sum_{j=1}^{M}\sum_{i=1,i\neq j}^{M}E\left(B_{2,i}\right)E\left(B_{2,j}\right)+2rE(B_2)\sum_{j=1}^{M}E\left(B_{2,j}\right)\right]\Big]$(4.6)

6. The mean number of customers in the queue when the server provides type 2 service is

$$L_{q2}=\lim_{z\to 1}\frac{d}{dz}\left[P^{(2,1)}(z)\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{(1-p)N_2(z)Q}{D(z)}\right]$$

Where, $N_2(z)=N_2(z)=B_{2,M}^{*}(s)\ldots..B_{2,1}^{*}(s)-1$

$L_{q2}=\frac{(1-p)Q[D'(1)N_2''(1)-N_2'(1)D''(1)]}{2\left(D'(1)\right)^2}$

$L_{q2}=\frac{(1-p)((1-q)\lambda)^2}{2(1-\rho)}\left[\{1-\lambda(1-q)p\sum_{j=1}^{N}E\left(B_{2,j}^{2}\right)\}+\{\sum_{j=1}^{M}E\left(B_{2,j}^{2}\right)+\sum_{j=1}^{M}\sum_{i=1,i\neq j}^{M}E\left(B_{2,i}\right)E\left(B_{2,j}\right)\}-\lambda(1-\right.$
$q)r(1-p)E(B_2)\{\sum_{j=1}^{M}E\left(B_{2,j}^{2}\right)+\sum_{j=1}^{M}\sum_{j=1,i\neq j}^{M}E\left(B_{2,i}\right)E\left(B_{2,j}\right)\}+\lambda(1-q)r(1-p)\{\sum_{j=1}^{M}E\left(B_{2,j}\right)\}\{E(B_2^2)+$
$\left.2E(B_2)\sum_{j=1}^{M}E\left(B_{2,j}\right)\}+\lambda p\{\sum_{j=1}^{M}E\left(B_{2,j}\right)\}\{\sum_{j=1}^{N}E\left(B_{1,j}^{2}\right)+\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}E\left(B_{1,i}\right)E\left(B_{1,j}\right)\}\right]$ (4.7)

7. The mean number of customers in the queue when the server provides an optional service is

$$L_{q3}=\lim_{z\to 1}\frac{d}{dz}\left[\sum_{j=1}^{M}P^{(2)}(z)\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{(1-p)N_3(z)Q}{D(z)}\right]$$

$$N_3(z) = rB_{2,M}^*(s)\ldots\ldots B_{2,1}^*(s)(B_2^*(s)-1)$$

$$L_{q3} = \frac{(1-p)Q[D'(1)N_3''(1) - N_3'(1)D''(1)]}{2\big(D'(1)\big)^2}$$

$$L_{q_3} = \frac{(1-p)((1-q)\lambda)^2}{2(1-\rho)}\Bigg[\{1-\lambda p\sum_{j=1}^{N}E(B_{1,j})\}\left\{2rE(B_2)\sum_{j=1}^{M}E(B_{2,j}) + rE(B_2^2)\right\} + \lambda(1-q)r(1-p)E(B_2)\left\{\sum_{j=1}^{M}E(B_{2,j}^2) + \sum_{j=1}^{M}\sum_{j=1,i\neq j}^{M}E(B_{2,i})E(B_{2,j})\right\} - \lambda(1-q)(1-p)\left\{\sum_{j=1}^{M}E(B_{2,j})\right\}\left\{E(B_2^2) + 2E(B_2)\sum_{j=1}^{M}E(B_{2,j})\right\} + \lambda(1-q)prE(B_2)\left\{\sum_{j=1}^{N}E(B_{1,j}^2) + \sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}E(B_{1,i})E(B_{1,j})\right\}\Bigg]$$

8.The idle probability is

$$Q + \sum_{j=1}^{N}P^{(1,j)} + \sum_{j=1}^{M}P^{(2,j)} + P^{(2)} = 1$$

$$Q = 1-\rho$$

Where, $\rho = \lambda(1-q)p\sum_{j=1}^{N}E(B_{1,j}) + \lambda(1-q)(1-p)\sum_{j=1}^{M}E(B_{2,j}) + \lambda(1-q)r(1-p)E(B_2)$

5 Numerical Illustrations

In this section, we provide some numerical illustrations related to the model discussed in this paper.For numerical calculations, we take

The service time distributions;

Type 1: $B_{1,j} = 1 - e^{\mu_{1,j}x}, j = 1,2,\ldots\ldots,N$

Type 2: $B_{2,j} = 1 - e^{\mu_{2,j}x}, j = 1,2,\ldots\ldots,M$

Optional:$B_2 = 1 - e^{\mu_2 x}$, Using the formula in section 4.

1.The mean number of customers in the queue is

$$L_q = \frac{((1-q)\lambda)^2 C_1}{2(1-\rho)}$$

Where, $C_1 = \frac{C_3}{\mu_{11}^2\mu_{12}^2\mu_{21}^2\mu_2^2}$

$$C_3 = (1-p)(2\mu_2^2\mu_{11}^2\mu_{12}^2 + 2r\mu_{21}^2\mu_{11}^2\mu_{12}^2 + 2r\mu_{21}\mu_2\mu_{11}^2\mu_{12}^2) + p(2\mu_{11}\mu_{12}\mu_{21}^2\mu_2^2 + 2\mu_{12}^2\mu_{21}^2\mu_2^2 + 2\mu_{11}^2\mu_{21}^2\mu_2^2)$$

$$\rho = \frac{\lambda(1-q)p\mu_{21}(\mu_{12}+\mu_{11})\mu_2 + \lambda(1-q)(1-p)\mu_{11}\mu_{12}\mu_2 + +\lambda(1-q)r(1-p)\mu_{11}\mu_{12}\mu_{21}}{\mu_{11}\mu_{12}\mu_{21}\mu_2}$$

2.The variance of thenumber of outcomes in the queue is

$$V_{L_q} = \frac{((1-q)\lambda)^2[2(1-\rho)(2\lambda(1-q)C_2 + 3C_1) + 3((1-q)\lambda)^2C_1^2]}{12((1-\rho))^2}$$

$$C_2 = \frac{C_4}{\mu_{11}^3\mu_{12}^3\mu_{21}^3\mu_2^3}$$

$$C_4 = (1-p)(6\mu_2^3\mu_{11}^3\mu_{12}^3 + 6r\mu_{21}^3\mu_{11}^3\mu_{12}^3 + 6r\mu_{21}\mu_2^2\mu_{11}^3\mu_{12}^3 + 6r\mu_2\mu_{21}^2\mu_{11}^3\mu_{12}^3) + p(\mu_{12}^3\mu_{21}^3\mu_2^3 + 6\mu_{11}^3\mu_{21}^3\mu_2^3 + 6\mu_{11}\mu_{12}^2\mu_{21}^3\mu_2^3 + 6\mu_{11}^2\mu_{12}\mu_{21}^3\mu_2^3)$$

3.The mean waiting time in the queue is

$$W_q = \frac{L_q}{\lambda}$$

4.The variance of the waiting time in the queue is

$$V_{W_q} = \frac{V_{L_q}}{\lambda^2}$$

5. The mean number of customers in the queue when the server provides type 1 service is

$$L_{q1} = \frac{((1-q)\lambda)^2pC_5}{2(1-\rho)\mu_{21}^2\mu_{12}^2\mu_{11}^2\mu_2^2}$$

$$C_5 = (2\mu_{12}^2 + \mu_{11}^2 + 2\mu_{11}\mu_{12})(\mu_{21}^2\mu_2^2) - \lambda(1-q)(1-p)(\mu_2 + r\mu_{21})(2\mu_{12}^2 + 2\mu_{11}^2 + 2\mu_{11}\mu_{12})(\mu_2\mu_{21}) + \lambda(1-q)(1-p)(\mu_{12} + \mu_{11})(2\mu_2^2 + 2\mu_{21}^2 + 2r\mu_{21}\mu_2)(\mu_{11}\mu_{12})$$

6.The mean number of customers in the queue when the server provides type2 service is

$$L_{q2} = \frac{((1-q)\lambda)^2(1-p)C_6}{2(1-\rho)\mu_{11}^2\mu_{21}^2\mu_{12}^2\mu_2^2}$$

$$C_6 = \mu_{11}^2\mu_{12}^2\mu_2^2 - \lambda(1-q)p\mu_{12}^2\mu_{11}\mu_2^2 - \lambda(1-q)p\mu_{11}^2\mu_{12}\mu_2^2 - \lambda(1-q)r(1-p)(\mu_{11}^2\mu_{12}^2\mu_2 + \lambda r(1-q)(1-p)(\mu_{21}\mu_{11}^2\mu_{12}^2 + \mu_2\mu_{11}^2\mu_{12}^2) + \lambda(1-q)p(\mu_{11}^2\mu_{21}\mu_2^2 + \mu_{12}^2\mu_{21}\mu_2^2 + \mu_{11}\mu_{12}\mu_{21}\mu_2^2)$$

7. The mean number of customers in the queue when the server provides an optional service is

$$L_{q3} = \frac{((1-q)\lambda)^2(1-p)C_7}{(1-\rho)\mu_{11}^2\mu_{21}^2\mu_{12}^2\mu_2^3}$$

$$C_7 = r\mu_{11}^2\mu_{12}^2\mu_{21} - \lambda(1-q)p\mu_{12}^2\mu_{11}\mu_{21} - \lambda(1-q)p\mu_{11}^2\mu_{12}\mu_{21}(\mu_2^2 + 2r\mu_{21}) + \lambda(1-q)r(1-p)(\mu_{11}^2\mu_{12}^2\mu_2\mu_2^2) + r(1-p)(\mu_{21}\mu_2\mu_{11}^2\mu_{12}^2 + \mu_{11}^2\mu_{12}^2\mu_2^2) + \lambda(1-q)pr(\mu_{12}^2\mu_2^2\mu_{21}^2 + \mu_{11}^2\mu_{21}^2\mu_2^2 + \mu_{11}\mu_{12}\mu_{21}^2\mu_2^2)$$

8.The idle probability is

$$\boldsymbol{Q = 1 - \rho}$$

Where,

$$\rho = \frac{\lambda(1-q)p\mu_{21}(\mu_{12} + \mu_{11})\mu_2 + \lambda(1-q)(1-p)\mu_{11}\mu_{12}\mu_2 + +\lambda(1-q)r(1-p)\mu_{11}\mu_{12}\mu_{21}}{\mu_{11}\mu_{12}\mu_{21}\mu_2}$$

Table 5.1: The mean number of customers in the queue:

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8
0.1	0.0001	0.0003	0.0004	0.0001	0.0003	0.0004	0.0002	0.0003	0.0004
0.2	0.0005	0.0011	0.0016	0.0005	0.0012	0.0017	0.0007	0.0013	0.0017
0.3	0.0010	0.0026	0.0038	0.0012	0.0027	0.0038	0.0015	0.0029	0.0039
0.4	0.0019	0.0047	0.0069	0.0021	0.0048	0.0070	0.0028	0.0052	0.0071
0.5	0.0030	0.0075	0.0111	0.0034	0.0077	0.0112	0.0044	0.0083	0.0114
0.6	0.0043	0.0110	0.0163	0.0049	0.0113	0.0165	0.0064	0.0123	0.0169
0.7	0.0060	0.0152	0.0228	0.0068	0.0157	0.0230	0.0089	0.0171	0.0236
0.8	0.0079	0.0202	0.0306	0.0090	0.0210	0.0309	0.0118	0.0228	0.0317
0.9	0.0101	0.0261	0.0397	0.0115	0.0271	0.0402	0.0152	0.0295	0.0413
1.0	0.0126	0.0329	0.0504	0.0144	0.0341	0.0510	0.0191	0.0373	0.0524

Table 5.2: The Variance of the number of customers in the queue:

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8
0.1	0.0001	0.0003	0.0004	0.0001	0.0003	0.0004	0.0002	0.0003	0.0004
0.2	0.0005	0.0012	0.0018	0.0005	0.0012	0.0018	0.0007	0.0013	0.0018
0.3	0.0011	0.0028	0.0042	0.0013	0.0029	0.0042	0.0016	0.0032	0.0043
0.4	0.0021	0.0053	0.0079	0.0023	0.0055	0.0080	0.0030	0.0059	0.0081
0.5	0.0033	0.0087	0.0130	0.0038	0.0090	0.0132	0.0049	0.0097	0.0135
0.6	0.0050	0.0132	0.0199	0.0056	0.0136	0.0201	0.0074	0.0147	0.0205
0.7	0.0070	0.0188	0.0287	0.0079	0.0194	0.0290	0.0104	0.0210	0.0297
0.8	0.0094	0.0258	0.0397	0.0107	0.0267	0.0401	0.0141	0.0289	0.0411
0.9	0.0123	0.0343	0.0533	0.0140	0.0355	0.0539	0.0185	0.0385	0.0553
1.0	0.0157	0.0445	0.0699	0.0179	0.0461	0.0706	0.0238	0.0501	0.0726

The mean number of customers in the queue, the variance of the number of customers in the queue, the mean waiting time in the queue, the variance of the waiting time in the queue, the mean number of customers in the queue when the server provides type 1 service, the mean number of customers in the queue when the server provides type 2 service and an optional service are calculated and tabulated.

Table 5.3: The Mean waiting time in the queue:

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8

0.1	0.0011	0.0028	0.0040	0.0013	0.0029	0.0041	0.0017	0.0031	0.0042
0.2	0.0023	0.0057	0.0082	0.0026	0.0058	0.0083	0.0034	0.0063	0.0085
0.3	0.0035	0.0086	0.0127	0.0040	0.0089	0.0128	0.0051	0.0096	0.0131
0.4	0.0047	0.0117	0.0173	0.0053	0.0121	0.0174	0.0069	0.0131	0.0178
0.5	0.0059	0.0149	0.0221	0.0067	0.0154	0.0223	0.0088	0.0167	0.0229
0.6	0.0072	0.0183	0.0272	0.0082	0.0189	0.0275	0.0107	0.0204	0.0282
0.7	0.0085	0.0217	0.0326	0.0097	0.0225	0.0329	0.0127	0.0244	0.0337
0.8	0.0098	0.0253	0.0382	0.0112	0.0262	0.0386	0.0148	0.0285	0.0396
0.9	0.0112	0.0290	0.0441	0.0128	0.0301	0.0446	0.0169	0.0328	0.0458
1.0	0.0126	0.0329	0.0504	0.0144	0.0341	0.0510	0.0191	0.0373	0.0524

Table 5.4: The Variance of the waiting time in the queue:

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8
0.1	0.0117	0.0287	0.0417	0.0132	0.0296	0.0420	0.0170	0.0317	0.0429
0.2	0.0121	0.0302	0.0441	0.0136	0.0311	0.0445	0.0176	0.0334	0.0454
0.3	0.0125	0.0317	0.0466	0.0141	0.0326	0.0470	0.0183	0.0351	0.0480
0.4	0.0129	0.0332	0.0493	0.0146	0.0343	0.0497	0.0190	0.0396	0.0509
0.5	0.0133	0.0349	0.0522	0.0151	0.0360	0.0527	0.0197	0.0388	0.0539
0.6	0.0138	0.0366	0.0552	0.0156	0.0378	0.0558	0.0204	0.0408	0.0571
0.7	0.0142	0.0384	0.0585	0.0162	0.0397	0.0591	0.0212	0.0429	0.0605
0.8	0.0147	0.0384	0.0585	0.0162	0.0397	0.0591	0.0212	0.0429	0.0605
0.9	0.0152	0.0424	0.0658	0.0173	0.0438	0.0665	0.0229	0.0476	0.0682
1.0	0.0157	0.0445	0.0699	0.0179	0.0461	0.0706	0.0238	0.0501	0.0726

Table 5.5: The Mean number of customers in the queue when the server provides type 1 service:

$\mu_{11} = 2$;$\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8
0.1	0.0000	0.0002	0.0004	0.0000	0.0002	0.0004	0.0000	0.0002	0.0004
0.2	0.0002	0.0010	0.0016	0.0002	0.0010	0.0016	0.0002	0.0010	0.0016
0.3	0.0005	0.0023	0.0037	0.0005	0.0023	0.0037	0.0005	0.0023	0.0037
0.4	0.0008	0.0041	0.0067	0.0008	0.0042	0.0067	0.0008	0.0042	0.0067
0.5	0.0013	0.0066	0.0108	0.0013	0.0067	0.0108	0.0013	0.0067	0.0108

0.6	0.0019	0.0098	0.0159	0.0019	0.0098	0.0159	0.0020	0.0099	0.0169
0.7	0.0027	0.0136	0.0222	0.0027	0.0137	0.0223	0.0028	0.0139	0.0224
0.8	0.0035	0.0182	0.0299	0.0036	0.0183	0.0300	0.0037	0.0187	0.0302
0.9	0.0046	0.0236	0.0389	0.0046	0.0238	0.0390	0.0048	0.0243	0.0394
1.0	0.0057	0.0299	0.0494	0.0058	0.0301	0.0496	0.0060	0.0308	0.0501

Table 5.6: The Mean number of customers in the queue when the server provides type 2 service :

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	$r = 0.2$			$r = 0.4$			$r = 0.9$		
	p=0.1	p=0.5	p=0.8	p=0.1	p=0.5	p=0.8	p=0.1	p=0.5	p=0.8
0.1	0.0001	0.0000	0.0000	0.0001	0.0000	0.0000	0.0001	0.0000	0.0000
0.2	0.0002	0.0001	0.0000	0.0002	0.0001	0.0000	0.0002	0.0001	0.0001
0.3	0.0005	0.0003	0.0001	0.0005	0.0003	0.0001	0.0005	0.0003	0.0001
0.4	0.0009	0.0005	0.0002	0.0009	0.0005	0.0002	0.0009	0.0005	0.0002
0.5	0.0014	0.0008	0.0004	0.0014	0.0008	0.0004	0.0014	0.0009	0.0004
0.6	0.0020	0.0013	0.0006	0.0020	0.0013	0.0006	0.0021	0.0013	0.0006
0.7	0.0028	0.0018	0.0008	0.0028	0.0018	0.0008	0.0029	0.0018	0.0008
0.8	0.0037	0.0024	0.0011	0.0037	0.0024	0.0011	0.0039	0.0025	0.0011
0.9	0.0047	0.0032	0.0014	0.0048	0.0032	0.0015	0.0050	0.0033	0.0015
1.0	0.0059	0.0040	0.0019	0.0060	0.0041	0.0019	0.0063	0.0042	0.0019

Table 5.7: The Mean number of customers in the queue when the server provides optional service :

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	$r = 0.2$			$r = 0.4$			$r = 0.9$		
	p=0.1	p=0.5	p=0.8	p=0.1	p=0.5	p=0.8	p=0.1	p=0.5	p=0.8
0.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
0.2	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0003	0.0001	0.0000
0.3	0.0001	0.0001	0.0000	0.0002	0.0001	0.0000	0.0006	0.0003	0.0001
0.4	0.0002	0.0001	0.0000	0.0004	0.0002	0.0001	0.0011	0.0006	0.0002
0.5	0.0003	0.0002	0.0001	0.0007	0.0004	0.0001	0.0017	0.0009	0.0003
0.6	0.0005	0.0003	0.0001	0.0010	0.0005	0.0002	0.0025	0.0013	0.0005
0.7	0.0007	0.0004	0.0001	0.0014	0.0008	0.0003	0.0035	0.0019	0.0007
0.8	0.0009	0.0005	0.0002	0.0018	0.0010	0.0004	0.0047	0.0025	0.0010
0.9	0.0011	0.0006	0.0003	0.0024	0.0013	0.0005	0.0060	0.0033	0.0013
1.0	0.0014	0.0008	0.0003	0.0030	0.0017	0.0007	0.0076	0.0043	0.0017

Table 5.8: The idle Probability :

$\mu_{11} = 2$; $\mu_{12} = 3$; $\mu_{21} = 4$; $\mu_2 = 5$;q=0.7

Λ	*r* = 0.2			*r* = 0.4			*r* = 0.9		
	p=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8	*p*=0.1	*p*=0.5	*p*=0.8
0.1	0.0103	0.0169	0.0217	0.0114	0.0175	0.0220	0.0141	0.0190	0.0226
0.2	0.0207	0.0337	0.0435	0.0228	0.0349	0.0440	0.0282	0.0379	0.0452
0.3	0.0310	0.0505	0.0652	0.0342	0.0523	0.0659	0.0423	0.0569	0.0677
0.4	0.0413	0.0674	0.0870	0.0456	0.0698	0.0879	0.0564	0.0758	0.0903
0.5	0.0516	0.0842	0.1087	0.0571	0.0873	0.1099	0.0705	0.0948	0.1129
0.6	0.0620	0.1011	0.1304	0.0685	0.1047	0.1319	0.0847	0.1137	0.1355
0.7	0.0723	0.1180	0.1522	0.0799	0.1222	0.1539	0.0988	0.1327	0.1581
0.8	0.0826	0.1348	0.1739	0.0913	0.1396	0.1758	0.1129	0.1516	0.1806
0.9	0.0930	0.1517	0.1957	0.1027	0.1571	0.1978	0.1270	0.1706	0.2032
1.0	0.1033	0.1685	0.2174	0.1141	0.1745	0.2198	0.1411	0.1895	0.2258

6 Conclusion

In this article, a single server non-Markovian queue with two types of phase service essential service. An optional service after second type service. Also the customer may balk. The supplementary variable technique has been utilized.The model can be extended by adding some more queueing characters..

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