

Numerical Solutions of Chen Dynamical System by Quintic B-Spline Method

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Abstract: - We propose a novel method based on quintic B-spline functions to obtain numerical solutions of the three-dimensional Chen chaotic system. The Chen system has applications in secure communication, modeling complex natural processes, and image encryption, due to its chaotic and complex dynamics. It can also be used to design and analyze control systems, optimize performance, and study physical phenomena through its mathematical model. Due to its various applications, solutions of this dynamical system are very important. Building on cubic B-spline formulations, the method yields dynamical properties that confirm its high efficiency for such models. Chaotic systems are sensitive to initial conditions. This physical behaviour is very well captured by the proposed numerical scheme. Computed results have been presented in tabular forms and figures. All computations have been carried out in 5.11 Dev C++. For numerical simulations, Matlab software has been used

Keywords: *Three-dimensional Chen chaotic system, Quintic B-spline function, Differential quadrature method.*

1. Introduction

Chaos is a very interesting and important subject in the study of nonlinear dynamical systems. Dynamical systems depend on initial conditions, and a very small perturbation produces very surprising outcomes. Several models exhibit chaos, which is a phenomenon shown by dynamical systems that depend sensitively on initial conditions. Swirling patterns of cigarette smoke, the fluttering of a flag in the wind, and similar phenomena demonstrate forms of chaotic behavior. The dynamical behavior of chaotic systems has been extensively studied by numerous researchers. Chaos theory finds applications across a wide range of fields, including population dynamics [1], cryptography, fluid mechanics [2], electrical systems, various branches of physics, lasers, celestial mechanics, weather forecasting [3], mechanical devices, electrical circuits [4], and many other areas of applied science. The Lorenz [5] in 1963 was the first one to study chaotic attractors. He had proposed a mathematical model of chaos.

Recently, significant efforts have been devoted to the development of chaotic and hyperchaotic dynamical models due to their wide range of applications. Various dynamical characteristics of these systems have been investigated using both theoretical and numerical approaches. In this study, we employ Quintic B-spline functions, and the resulting analysis shows that the proposed method is highly effective for solving such three-dimensional models. We have analyzed the numerical solutions of the Chen system in the present computational study. The system is based on the quintic B-spline method. The governing equations of the Chen system are given by

$$\frac{dx}{dt} = a(y - x), \quad (1.1)$$

$$\frac{dy}{dt} = (c - a)x - xz + cy, \quad (1.2)$$

$$\frac{dz}{dt} = xy - bz, \quad (1.3)$$

where a , b , and c are positive constants. We have both chaotic and nonchaotic behavior of the Chen system, also it has been well established that the Chen system is dual to the Lorenz system in the sense that in the Lorenz system.

Exact solutions for such systems are often difficult to find. Consequently, researchers have focused on finding solutions to these types of ordinary differential equations using various numerical and semi-analytical methods. Lozi and Alexander [6] introduced a novel approach based on a modified power series to solve the Chen attractor numerically. Allan [7] applied the homotopy analysis method to illustrate the chaotic behavior of dynamical systems. Mossa et al. [8] developed a technique based on differential transformation to obtain numerical solutions for the Chen dynamical system. Chowdhury et al. [9] employed the homotopy perturbation method to obtain solutions for the Lorenz and Chen systems, respectively. It has been observed that both the homotopy analysis method and the Adomian decomposition method involve lengthy computations due to the evaluation of homotopy and Adomian polynomials. Mittal and Dahiya [10] applied this approach to the Kuramoto–Sivashinsky equation. Eftekhari and Jafari [11] utilized the Lagrange polynomial approach to compute numerical solutions of chaotic dynamical systems. Among these techniques, the variational iteration method is comparatively simple; however, it suffers from long computational times because of the presence of exponential terms in its iterative process. Furthermore, as noted in [9], the homotopy perturbation method provides accurate solutions only over a short time interval.

To address the limitations of the homotopy method, Chowdhury et al. proposed an approach in which the entire time interval is first divided into subintervals, and the homotopy method is then applied independently on each subinterval. However, this approach still involves cumbersome calculations for evaluating polynomial coefficients. Similarly, in the Lagrange polynomial method, the coefficients vary across subintervals and must be computed separately for each, resulting in increased computational time. In our approach, the coefficients of the differential quadrature do not need to be determined for each subinterval, thereby reducing computational time. In recent years, differential quadrature has emerged as a highly efficient numerical discretization technique for obtaining accurate solutions to ordinary and partial differential equations. The quintic B-spline method has been widely employed for solving nonlinear differential equations and for the accurate numerical solutions of various mathematical models. Iqbal et al. [12] investigated a second-order coupled nonlinear Schrodinger equation using the quintic B-spline method. Mohammadi [13] utilized the quintic B-spline collocation method to solve the Black–Scholes equation. Furthermore, Mirzaee and Alipour [14] proposed a numerical scheme based on quintic B-splines for n -dimensional stochastic Ito–Volterra integro–differential equations. Chandrasekharan et al. [15] also examined related nonlinear problems using the quintic B-spline technique. Zaki [16] has formulated Quintic B-spline method to solve KdVB equation. Kaur and Joshi [17] presented the coupled Korteweg–de Vries equation by using quintic method of quadrature analysis based on hyperbolic B-spline. Saka et al. [18] have examined a regularized long wave equation using quintic B-spline collocation method. Ren et al. [19] proposed the quintic B-spline collocation method to find a Bona–Smith family of the Boussinesq equation. The rest of the paper is structured as follows: In Sec. 2, the detailed explanation of quintic B-splines is presented, followed by the evaluation of the weighting coefficients for the differential quadrature method. Section 3 describes the implementation of the proposed method for dynamical systems. In Section 4, the application of the method to the Chen system is discussed, and the corresponding numerical results are presented. Section 5 presents the application of our method to several important chaotic dynamical systems, along with an explanation of the systems' physical behavior.

2. Differential quadrature method and Quintic B-splines

The basic concept of the differential quadrature method (DQM) was introduced by Bellman and his colleagues in the 1970s. It is a numerical discretization technique used for solving well-posed ordinary and partial differential equations. In DQM, the derivative of an unknown function is approximated as a weighted linear combination of the function values at selected discrete points within the domain. Let $[a, b]$ be the domain of interest. We divide it uniformly by choosing knots t_i , $i = 1, 2, \dots, N$, such that the spacing between two consecutive points is constant,

i.e. $t_i - t_{i-1} = h$. Using this setup, the r -th order derivative of a function f at any knot can be approximated by a differential quadrature formula.

$$f^r(t_j) = \sum_{k=0}^n a^r_{jk} f(t_k), \quad j = 0, 1, 2, \dots, n \quad (2.1)$$

where a^r_{jk} represents the r -th order weighting coefficient and $f^{(r)}(t)$ is the r -th order derivative of f with respect to t . Quintic B-splines: The quintic B-spline functions $Q_j(t)$ are defined as follows:

$$Q_j(t) = \frac{1}{h^5} \begin{cases} (t - t_{j-3})^5, & t \in [t_{j-3}, t_{j-2}), \\ (t - t_{j-3})^5 - 6(t - t_{j-2})^5, & t \in [t_{j-2}, t_{j-1}), \\ (t - t_{j-3})^5 - 6(t - t_{j-2})^5 + 15(t - t_{j-1})^5, & t \in [t_{j-1}, t_j), \\ (t_{j+3} - t)^5 - 6(t_{j+2} - t)^5 + 15(t_{j+1} - t)^5, & t \in [t_j, t_{j+1}), \\ (t_{j+3} - t)^5 - 6(t_{j+2} - t)^5, & t \in [t_{j+1}, t_{j+2}), \\ (t_{j+3} - t)^5, & t \in [t_{j+2}, t_{j+3}), \\ 0, & \text{otherwise} \end{cases}$$

t	t_{j-3}	t_{j-2}	t_{j-1}	t_j	t_{j+1}	t_{j+2}	t_{j+3}
$Q_j(t)$	0	1	26	66	26	1	0
$Q_j^t(t)$	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$-\frac{50}{h}$	$-\frac{5}{h}$	0
$Q_j^{tt}(t)$	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$-\frac{120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
$Q_j^{ttt}(t)$	0	$\frac{60}{h^3}$	$-\frac{120}{h^3}$	0	$\frac{120}{h^3}$	$-\frac{60}{h^3}$	0

Table 1: Values of $Q_m(t)$ and its derivatives at the knots.

Where $h = t_i - t_{i-1}$ for all $j = -1, 0, 1, \dots, N+1, N+2$, and $Q_{-1}, Q_0, Q_1, \dots, Q_{N+1}, Q_{N+2}$ form a basis over the region $a \leq t \leq b$.

Using these basis functions, the B-splines and their first five derivatives at the nodes can be obtained as listed in Table 1. A spline is a polynomial function defined piecewise to interpolate given data points. Splines are valued for their smoothness, providing curves that are both continuous and differentiable. B-splines, in particular, are generated through a set of basis functions that determine their overall form and smoothness. These functions are polynomials defined over separate intervals, which are combined together to form a smooth overall curve.

Evaluation of weighting coefficients

Thomas algorithm gives the weighting coefficients of $a^{(1)}_{1,-1}, a^{(1)}_{1,0}, a^{(1)}_{1,1}, a^{(1)}_{1,2}, \dots, a^{(1)}_{1,N}, a^{(1)}_{1,N+1}, a^{(1)}_{1,N+2}$. In the same way, we can also find out the weighting coefficients for $j = -1, 0, 1, 2, \dots, N, N+1, N+2$ Using these coefficients, we are able to find the first-order derivative.

4. Numerical Experiments and Results

The Quintic B-spline differential quadrature method is employed to analyze five well-known chaotic dynamical systems. The numerical solutions obtained from this approach are illustrated graphically. The butterfly effect is demonstrated through these graphical results. Furthermore, the relative errors are evaluated using the following expression.

The Chen System

We consider equations (1.1) – (1.3) in the domain $0 \leq t \leq 2.5$ with the initial conditions

$$x(0) = -10, \quad y(0) = 0, \quad z(0) = 37.$$

For this system, we investigate numerical solutions corresponding to both chaotic and nonchaotic cases. For the numerical simulations, the parameters are chosen as $n = 10$, $\Delta t = 0.001$, $a = 35$, and $c = 28$. For the non-chaotic case, the parameter is taken as $b = 12$, whereas for the chaotic case, $b = 3$. In Figs 1-3., the numerical values of x , y , and z are plotted for the non-chaotic Chen system.

Phase portraits in the x - y , x - z , and y - z planes are shown in Figs. 4–6. The chaotic behavior of the Chen system for the state variables x , y , and z is illustrated in Figs. 7–9. The corresponding phase portraits are presented in Figs. 10-11.

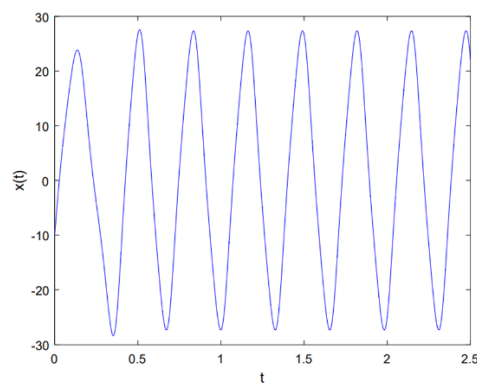


Figure1: Plot of the numerical solution x for the Chen system

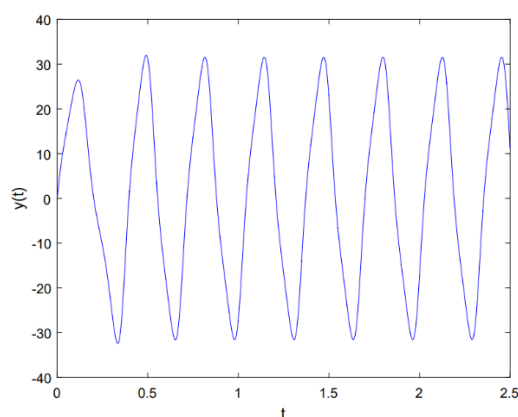


Figure 2: Plot of the numerical solution y for the Chen system

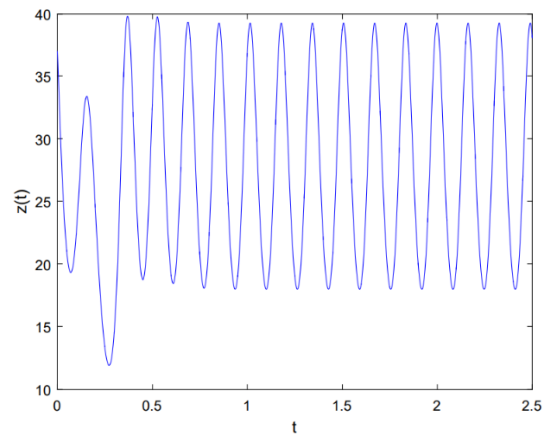


Figure 3: Plot of the numerical solution z for the Chen system

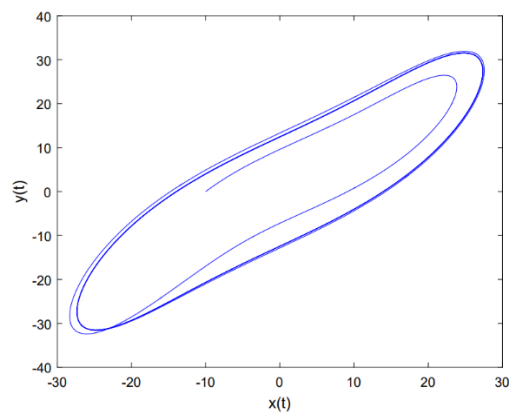


Figure 4: Phase portrait for the non-chaotic Chen system

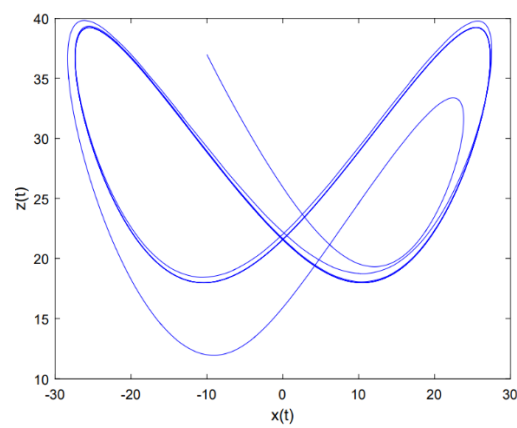


Figure 5: Phase portrait for the non-chaotic Chen system

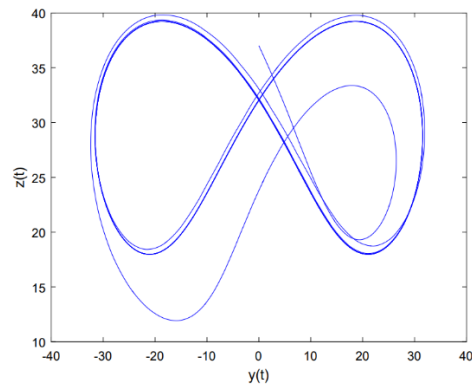


Figure 6: Phase portrait for the non-chaotic Chen system

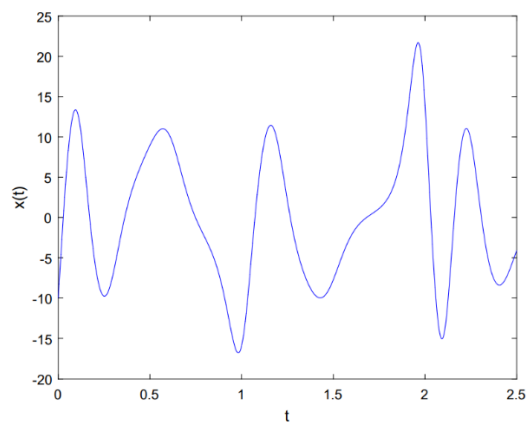


Figure 7: Plot of x for the chaotic Chen system

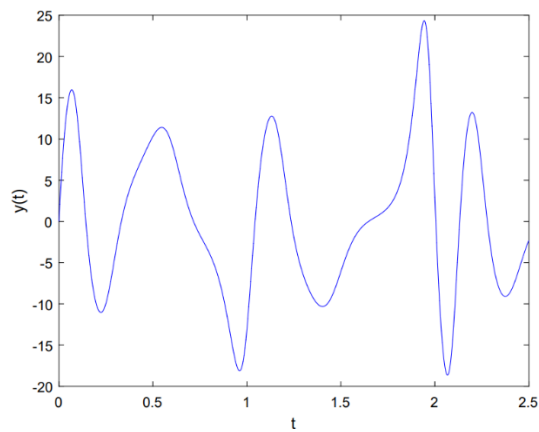


Figure 8: Plot of y for the chaotic Chen system

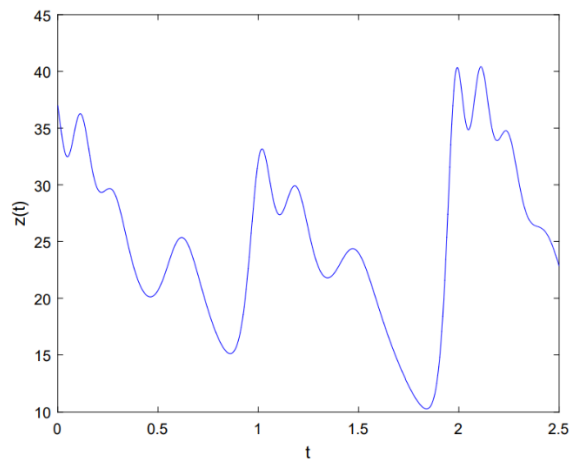


Figure 9: Plot of z for the chaotic Chen system

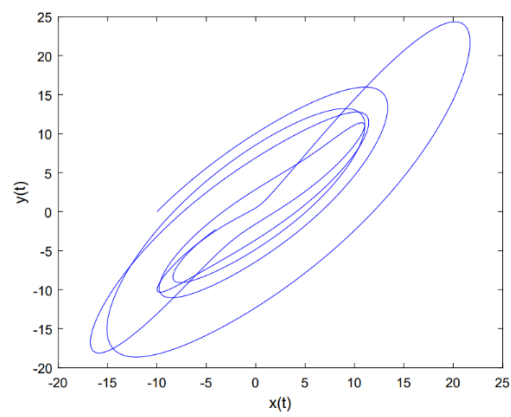


Figure 10: Phase portrait for the chaotic Chen system.

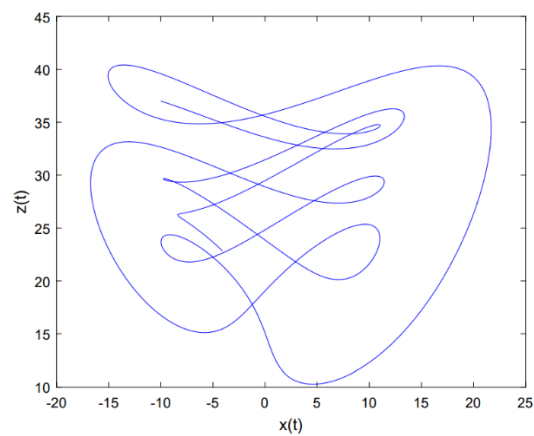


Figure 11: Phase portrait for the chaotic Chen system

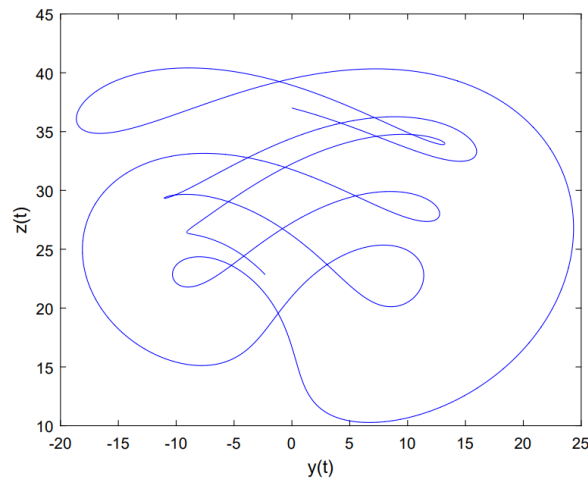


Figure 12: Phase portrait for the chaotic Chen system.

t	DQM method	RK-4 method
0.50	2.10E-04	5.07E-02
1.00	3.04E-05	2.23E-03
1.50	8.01E-06	1.30E-03
2.00	1.80E-06	1.00E-03

Comparison of relative errors for the Chen system.

5. Conclusion

In this work, the chaotic dynamical system Chen system has been studied using Quintic B-spline functions. This method offers the advantage of eliminating the need to compute weighting coefficients at each time interval, thereby reducing computational cost. As a result, the overall processing time is significantly decreased. The computed solutions are presented graphically and show good agreement with results available in the literature. Moreover, the numerical solutions obtained using the cubic B-spline method accurately capture the expected physical behavior of three-dimensional dynamical models. It can therefore be concluded that the cubic B-spline-based numerical technique is an efficient and reliable tool for solving such dynamical systems.

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