Stability of a Functional Equation Involving Mean Sum of Consecutive Varibales with Quadratic Property

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Abstract:- In, this paper we establish Ulam-Hyers stability of quadratic functional equation involving Mean sum of functions of consecutive variables of the form

$$\sum_{a=1}^{m} q\left(\frac{1}{a}\sum_{b=1}^{a} u_b\right) = \sum_{a=1}^{m}\left(\sum_{b=a}^{m}\frac{1}{b^2}\right)q(u_a) + \frac{1}{2}\sum_{a=1}^{m}\left(\sum_{b=a}^{m}\frac{1}{b^2}\left(q(u_b+u_a)-q(u_b-u_a)\right)\right)$$

in Banach space via direct and fixed point method.

Keywords*:* Quadratic functional equation, Ulam-Hyers-Rassias Stability of functional equation, Cauchy Sequence, Banach Space, Fixed point, Hyers Method, Contraction Mapping.

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1. Introduction

The stability problem of functional equations was raised by Ulam S M. [32] in 1964. In fact he imposed the question “Suppose that a function fulfilling a functional equation approximately according to some concord. Is it then possible to find near this function a function satisfying the equation accurately?” In 1941 Hyers D H [16] gave a consequential partial solution to this problem in his paper.

Hyers’ result was generalized by Aoki.T [2] for additive mappings. In 1978, Rassias Th M [29] generalized Hyers’ result, a fact which rekindled interest in the field. Such type of stability is now called the Ulam-Hyers-Rassias stability of functional equations.We refer the curious readers for further information on such problems to, for example, [4,5,6,7,9,10,13,14,17,18,22,23].

The main aim of this paper is to give the generalized Ulam-Hyers-Rassias stability of quadratic functional equation of the form

$$\sum_{a=1}^{m} q\left(\frac{1}{a}\sum_{b=1}^{a} u_b\right) = \sum_{a=1}^{m}\left(\sum_{b=a}^{m}\frac{1}{b^2}\right)q(u_a) + \frac{1}{2}\sum_{a=1}^{m}\left(\sum_{b=a}^{m}\frac{1}{b^2}\left(q(u_b+u_a)-q(u_b-u_a)\right)\right) \tag{1}$$

in Banach space using direct and fixed point method.

Now, we recall the fundamental result of alternative fixed point theory.

Theorem. 1.1. [The alternative fixed point] Suppose for a complete generalized metric space (ψ,d) and strictly contractive mapping $\xi:\psi\rightarrow\psi$ with Lipschitz constant L . Then for each given $u\in\psi$,either $d(\xi^n u,\xi^{n+1}u)=\infty$ for all $n\geq 0$,

or there exist a natural number n_0 such that

(AF1) $d\left(\xi^n u, \xi^{n+1} u\right) < \infty$ for all $n \geq n_o$;

(AF2) The sequence $\{\xi^n u\}$ is convergent to a fixed point u^* of ξ;

(AF3) u^* is the unique fixed point ξ in the set $\varpi = \{u \in \psi : d\left(u, \xi^{n_o} u\right) < \infty\}$;

(AF4) $d\left(u, u^*\right) \leq \frac{1}{1-L} d(u, \xi u)$ for all $u \in \psi$

2.STABILITY RESULT IN BANACH SPACE USING DIRECT METHOD

In this section we analyze the Ulam-Hyers stability of functional equation (1) in Banach space using direct method. Hereafter throughout this paper, assume that ψ_1 be a normed linear space and ψ_2 be a Banach space. We use the following abbreviation for a given function $q : \psi_1 \to \psi_2$ by

$$\partial q(u_1, u_2, \ldots, u_m) = \sum_{a=1}^{m} q\left(\frac{1}{a} \sum_{b=1}^{a} u_b\right) - \sum_{a=1}^{m}\left(\sum_{b=a}^{m} \frac{1}{b^2}\right) q(u_a) - \frac{1}{2} \sum_{a=1}^{m}\left(\sum_{b=a}^{m} \frac{1}{b^2}\left(q(u_b + u_a) - q(u_b - u_a)\right)\right)$$

for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$.

Theorem 2.1 Let $\tau = \pm 1$ be fixed. Also let $\gamma : \psi_1{}^m \to [0, \infty)$ be a mapping satisfies

$$\lim_{n \to \infty} \frac{\gamma\left(m^{\tau n} u_1, m^{\tau n} u_2, \ldots, m^{\tau n} u_m\right)}{m^{2n\tau}} = 0 \tag{2}$$

for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$ and $q : \psi_1 \to \psi_2$ be a mapping satisfies the inequality

$$\left\|\partial q(u_1, u_2, \ldots, u_m)\right\| \leq \gamma(u_1, u_2, \ldots, u_m) \tag{3}$$

for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$. Then there exists a unique quadratic mapping $Q : \psi_1 \to \psi_2$ such that

$$\left\|Q(u) - q(u)\right\| \leq \sum_{k=\frac{1-\tau}{2}}^{\infty} \frac{\alpha\left(m^{\tau k} u\right)}{m^{2k\tau}} \tag{4}$$

for all $u \in \psi_1$, where the mapping $Q(u)$ and $\alpha(u)$ are defined by $Q(u) = \lim_{k \to \infty} \frac{q\left(m^{\tau k} u\right)}{m^{2k\tau}}$ and $\alpha(u) = \gamma(0,0,\ldots,mu)$ for all $u \in \psi_1$ respectively.

Proof. **Case (i)**: Assume $\tau = 1$. Replacing $(u_1, u_2, \ldots, u_m)$ by $(0,0,\ldots,u)$ in (3), we get

$$\left\|q\left(\frac{u}{m}\right) - \frac{q(u)}{m^2}\right\| \leq \gamma(0,0,\ldots,u) \tag{5}$$

for all $u \in \psi_1$. Changing u by mu in (5), we obtain

$$\left\|q(u) - \frac{q(mu)}{m^2}\right\| \leq \gamma(0,0,\ldots,u) \tag{6}$$

for all $u \in \psi_1$. Taking $\gamma(0,0,\ldots,mu) = \alpha(u)$ into (6), we arrive

$$\left\| q(u) - \frac{q(mu)}{m^2} \right\| \le \alpha(u) \tag{7}$$

for all $u \in \psi_1$. In general

$$\left\| q(u) - \frac{q(m^n u)}{m^{2n}} \right\| \le \sum_{k=0}^{n-1} \frac{\alpha(m^k u)}{m^{2k}} \tag{8}$$

for all $u \in \psi_1$. Setting u by $m^l u$ in (8) and dividing m^{2l}, we reach

$$\left\| \frac{q(m^l)}{m^{2l}} - \frac{q(m^{n+l} u)}{m^{2(n+l)}} \right\| \le \sum_{k=0}^{n-1} \frac{\alpha(m^{k+l} u)}{m^{2(k+l)}} \tag{9}$$

for all $u \in \psi_1$. As limit $l \to \infty$ the right off (9) approaches to 0. Hence the sequence $\left\{ \frac{q(m^n u)}{m^{2n}} \right\}$ is a Cauchy sequence in ψ_2. Since ψ_2 is Banach, therefore the sequence $\left\{ \frac{q(m^n u)}{m^{2n}} \right\}$ converges to some point $Q(u) \in \psi_2$. So, we define

$$Q(u) = \lim_{n \to \infty} \frac{q(m^n u)}{m^{2n}} \tag{10}$$

for all $u \in \psi_1$. To prove that Q satisfies (1), replacing $(u_1, u_2, ..., u_m)$ by $(m^n u_1, m^n u_2, ..., m^n u_m)$ and dividing m^{2n} in (3) and allowing limit $n \to \infty$ and using (10), it is easy to see that Q satisfies (1) for all $u_i \in \psi_1, (i = 1,2,3,...,m)$. To prove Q is unique, we let Q' be another mapping satisfies (1) and (3).Then we have

$$\begin{aligned}
\|Q(u) - Q'(u)\| &= \frac{1}{m^{2n}} \|Q(m^n u) - Q'(m^n u)\| \\
&\le \frac{1}{m^{2n}} \|Q(m^n u) - q(m^n u)\| + \frac{1}{m^{2n}} \|q(m^n u) - Q'(m^n u)\| \\
&\le 2 \sum_{k=0}^{\infty} \frac{\alpha(m^{k+n} u)}{m^{2(k+n)}} \quad \to 0 \text{ as } n \to 0
\end{aligned}$$

for all $u \in \psi_1$. Therefore $Q(u) = Q'(u)$ which proves the uniqueness.

Case (ii): For $\tau = -1$. Putting u by $\frac{u}{m}$ in (7) and multiplying m^2, we reach

$$\left\| m^2 q\left(\frac{u}{m}\right) - q(u) \right\| \le m^2 \alpha\left(\frac{u}{m}\right) \tag{11}$$

for all $u \in \psi_1$. In general

$$\left\| m^{2n} q\left(\frac{u}{m^n}\right) - q(u) \right\| \le \sum_{k=1}^{n-1} m^{2k} q\left(\frac{u}{m^k}\right) \tag{12}$$

for all $u \in \psi_1$. The rest of the proof is similar to that of **Case (i).** This completes the proof of the Theorem.

The following corollary is the immediate consequence of the Theorem II.1 concerning the stability of (1).

Corollary 2.2 Assume ε be positive number and ω be real number with $\omega \neq 2$. Let $q:\psi_1 \to \psi_2$ be a mapping satisfying the functional inequality

$$\left\| \partial q(u_1, u_2, \ldots, u_m) \right\| \le \begin{cases} \varepsilon; \\ \varepsilon \sum_{i=1}^{m} \| u_i \|^{\omega}; \omega \neq 2 \end{cases} \tag{13}$$

for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$. Then there exists unique quadratic mapping $Q:\psi_1 \to \psi_2$ such that

$$\left\| Q(u) - q(u) \right\| \le \begin{cases} \dfrac{\varepsilon m^2}{\left| m^2 - 1 \right|}; \\ \dfrac{\varepsilon m^{2+\omega} \|u\|^{\omega}}{\left| m^{\omega} - m^2 \right|}; \omega \neq 2 \end{cases} \tag{14}$$

for all $u \in \psi_1$.

3.STABILITY RESULT IN BANACH SPACE USING FIXED POINT METHOD

In this section we analyze the Ulam-Hyers stability of functional equation (1) in Banach space using fixed point method.

Theorem 3.1 Suppose $q:\psi_1 \to \psi_2$ be a function satisfies the inequality (3) for which there exists a mapping $\gamma:{\psi_1}^m \to [0,\infty)$ with the condition

$$\lim_{n\to\infty} \frac{\gamma\left(v_i^{\,n} u_1, v_i^{\,n} u_2, \ldots, v_i^{\,n} u_m\right)}{v_i^{\,n}} = 0, \text{ where } v_i = \begin{cases} \dfrac{1}{m}, i = 0 \\ m, i = 1 \end{cases} \tag{15}$$

for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$. If there exists $L = L(i)$ such that the function $u \to \alpha(u)$ has the properties

$$\alpha(u) = \gamma(0,0,\ldots,mu); \quad L\alpha(u) = \frac{1}{v_i^{\,2}} \alpha(v_i u) \tag{16}$$

for all $u \in \psi_1$.Then there exists unique quadratic function $Q:\psi_1 \to \psi_2$ satisfies the functional equation (1) and the inequality

$$\left\| Q(u) - q(u) \right\| \le \frac{L^{1-i}}{1-L} \alpha(u) \tag{17}$$

for all $u \in \psi_1$.

Proof. Assume a set

$$\Phi = \{ q \mid q:\psi_1 \to \psi_2, q(0) = 0 \} \tag{18}$$

and define the generalized metric on Φ as

$$d(q,q') = \inf\{\sigma > 0; \|q(u) - q'(u)\| \le \sigma\alpha(u)\} \tag{19}$$

for all $u \in \psi_1$, it is easy to see that (Φ, d) is complete.

Define a mapping $\xi_q : \Phi \to \Phi$ by

$$\xi_q(u) = \frac{q(v_i u)}{{v_i}^2} \tag{20}$$

for all $u \in \psi_1$. Now to claim ξ_q is contraction on Φ, if for any $q, q' \in \Phi$ such that

$$d(q,q') \le \sigma$$

$$\Rightarrow \|q(u) - q'(u)\| \le \sigma\alpha(u)$$

$$\Rightarrow \left\|\frac{q(uv_i)}{{v_i}^2} - \frac{q'(uv_i)}{{v_i}^2}\right\| \le \sigma\frac{\alpha(uv_i)}{{v_i}^2}$$

$$\Rightarrow \|\xi_q(u) - \xi_{q'}(u)\| \le \sigma L\alpha(u)$$

$$\Rightarrow d(\xi_q, \xi_{q'}) \le \sigma L$$

$$\Rightarrow d(\xi_q, \xi_{q'}) \le Ld(q,q') \tag{21}$$

This shows that the mapping $\xi_q : \Phi \to \Phi$ is contractive mapping on Φ with Lipschitz constant L. It's trail around with (7) , we have

$$\left\|q(u) - \frac{q(mu)}{m^2}\right\| \le \alpha(u) \tag{22}$$

for all $u \in \psi_1$. With help of (19) , (20) and (22) it is easy to see that

$$d(\xi_q, q) \le 1 \text{ , for } i = 1$$

$$\Rightarrow d(\xi_q, q) \le L^{1-i} \tag{23}$$

Similarly,from (11) for the case $i = 0$, it is easy to see that

$$\left\|m^2 q\left(\frac{u}{m}\right) - q(u)\right\| \le L\alpha(u) \tag{24}$$

for all $u \in \psi_1$,where $L\alpha(u) = m^2\alpha\left(\frac{u}{m}\right)$.With help of (19) ,(20) and (24), we arrive

$$d(\xi_q, q) \le L \text{ , for } i = 0$$

$$\Rightarrow d(\xi_q, q) \le L^{1-i} \tag{25}$$

Combining (23) and (25), we have

$$d(\xi_q, q) \le L^{1-i} < \infty \tag{26}$$

Therefore (AF1) of Theorem I.1 holds.

By (AF2) of Theorem I.1, there exists a fixed point Q of ξ_q in Φ such that

$$Q(u)=\lim_{n\to\infty}\frac{q\left(v_i^{\ n}u\right)}{v_i^{\ 2n}}\text{, for all } u\in\psi_1.$$

To show that $Q:\psi_1\to\psi_2$ satisfies the functional equation (1), idea of this just similar lines to that of Theorem II.1.

Again by (AF3) of Theorem I.1, $Q(u)$ is the unique fixed point of ξ_q in the set

$$\aleph=\{Q(u)\in\Phi:d\left(Q(u),q(u)\right)<\infty\}.$$

Finally by (AF4) of the Theorem I.1, we arrive

$$d(Q,q)\le\frac{L^{1-i}}{1-L}d\left(\xi_q,q\right)\Rightarrow d(Q,q)\le\frac{L^{1-i}}{1-L}\alpha(u)$$

$$\Rightarrow\|Q(u)-q(u)\|\le\frac{L^{1-i}}{1-L}\alpha(u)$$

for all $u\in\psi_1$. Hence the proof is complete.

Corollary 3.2 Assume ε be positive number and ω be real number with $\omega\neq2$. Let $q:\psi_1\to\psi_2$ be a mapping satisfying the functional inequality

$$\left\|\partial q\left(u_1,u_2,\ldots,u_m\right)\right\|\le\begin{cases}\varepsilon;\\ \varepsilon\sum_{i=1}^{m}\|u_i\|^{\omega};\omega\neq2\end{cases}\tag{27}$$

for all $u_i\in\psi_1,(i=1,2,3,\ldots,m)$. Then there exists unique quadratic mapping $Q:\psi_1\to\psi_2$ such that

$$\|Q(u)-q(u)\|\le\begin{cases}\dfrac{\varepsilon m^2}{\left|m^2-1\right|};\\ \dfrac{\varepsilon m^{2+\omega}\|u\|^{\omega}}{\left|m^{\omega}-m^2\right|};\omega\neq2\end{cases}\tag{28}$$

for all $u\in\psi_1$.

Proof. Let us take

$$\gamma\left(u_1,u_2,\ldots,u_m\right)=\begin{cases}\varepsilon;\\ \varepsilon\sum_{i=1}^{m}\|u_i\|^{\omega};\end{cases}\tag{29}$$

for all $u_i\in\psi_1,(i=1,2,3,\ldots,m)$ in Theorem III.1. Replacing $\left(u_1,u_2,\ldots,u_m\right)$ by $\left(m^nu_1,m^nu_2,\ldots,m^nu_m\right)$ and dividing by m^{2n} in the equation (29), we reach

$$\frac{1}{{v_i}^n}\gamma\left({v_i}^n u_1, {v_i}^n u_2, \ldots, {v_i}^n u_m\right) = \begin{cases} \dfrac{\varepsilon}{{v_i}^n}; \\ \dfrac{\varepsilon}{{v_i}^n}\sum_{i=1}^{m} {v_i}^{\omega n} \| u_i \|^{\omega}; \end{cases}$$

$$= \begin{cases} \to 0 \\ \to 0 \end{cases} \text{as } n \to 0.$$

Therefore (15) holds for all $u_i \in \psi_1, (i = 1,2,3,\ldots,m)$. Now it follows from (16), we have

$$\alpha(u) = \gamma(0,0,\ldots,mu) = \begin{cases} \varepsilon; \\ \varepsilon m^{\omega}\|u\|^{\omega}; \end{cases}$$

and

$$\frac{1}{{v_i}^2}\alpha(v_i u) = \begin{cases} \dfrac{1}{{v_i}^2}\varepsilon; \\ \dfrac{1}{{v_i}^2}{v_i}^{\omega}\varepsilon m^{\omega}\|u\|^{\omega}; \end{cases}$$

$$\Rightarrow \frac{1}{{v_i}^2}\alpha(v_i u) = \begin{cases} {v_i}^{-2}\varepsilon; \\ {v_i}^{\omega-2}\varepsilon m^{\omega}\|u\|^{\omega}; \end{cases}$$

$$\Rightarrow \frac{1}{{v_i}^2}\alpha(v_i u) = \begin{cases} {v_i}^{-2}\alpha(u); \\ {v_i}^{\omega-2}\alpha(u); \end{cases}$$

$$\Rightarrow \frac{1}{{v_i}^2}\alpha(v_i u) = \begin{cases} L\alpha(u); \\ L\alpha(u); \end{cases}$$

for all $u \in \psi_1$.

Case (i): For $i = 0$, we have $L = {v_i}^{-2} = \left(\frac{1}{m}\right)^{-2} = m^2$. From (17), we obtain

$$\|Q(u) - q(u)\| \le \frac{L^{1-i}}{1-L}\alpha(u) = \frac{L}{1-L}\alpha(u) = \frac{\varepsilon m^2}{1-m^2} \tag{30}$$

for all $u \in \psi_1$.

Case (ii): For $i = 1$, we have $L = {v_i}^{-2} = (m)^{-2} = \frac{1}{m^2}$. From (17), we obtain

$$\|Q(u) - q(u)\| \le \frac{L^{1-i}}{1-L}\alpha(u) = \frac{1}{1-L}\alpha(u) = \frac{\varepsilon m^2}{1-m^2} \tag{31}$$

for all $u \in \psi_1$.

Case (iii): For $i = 0$, we have $L = {v_i}^{\omega-2} = \left(\frac{1}{m}\right)^{\omega-2} = \frac{m^2}{m^{\omega}}$. From (17), we obtain

$$\|Q(u)-q(u)\| \le \frac{L^{1-i}}{1-L}\alpha(u) = \frac{L}{1-L}\alpha(u) = \frac{\varepsilon m^{\omega+2}\|u\|^{\omega}}{m^{\omega}-m^{2}} \tag{32}$$

for all $u \in \psi_1$.

Case (iv): For $i=1$, we have $L = \nu_i^{\omega-2} = (m)^{\omega-2} = \frac{m^{\omega}}{m^2}$. From (17), we obtain

$$\|Q(u)-q(u)\| \le \frac{L^{1-i}}{1-L}\alpha(u) = \frac{1}{1-L}\alpha(u) = \frac{\varepsilon m^{\omega+2}\|u\|^{\omega}}{m^{2}-m^{\omega}} \tag{33}$$

for all $u \in \psi_1$. This competes the proof of the corollary.

4. Some real life application problems

In this section we provide some real life application problems based on the quadratic functional equation (1) for $m=2$ and $m=3$. For $m=2$ and $m=3$, the functional equation (1), becomes

$$\begin{aligned} & q(u_1)+q\left(\frac{u_1+u_2}{2}\right) \\ & =\left(1+\frac{1}{2^2}\right)q(u_1)+\frac{1}{2^2}q(u_2)+\frac{1}{2}\left[\left(1+\frac{1}{2^2}\right)\left(q(u_1+u_2)-q(u_1-u_2)\right)\right]; \end{aligned} \tag{34}$$

and

$$\begin{aligned} & q(u_1)+q\left(\frac{u_1+u_2}{2}\right)+q\left(\frac{u_1+u_2+u_3}{3}\right) \\ & =\left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)q(u_1)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)q(u_2)+\left(\frac{1}{3^2}\right)q(u_3)+\frac{1}{2}\begin{bmatrix}\left(\frac{1}{2^2}+\frac{1}{3^2}\right)\left(q(u_1+u_2)-q(u_1-u_2)\right) \\ +\frac{1}{3^2}\begin{pmatrix} q(u_1+u_2)-q(u_1-u_2) \\ +q(u_2+u_3)-q(u_2-u_3)\end{pmatrix}\end{bmatrix}. \end{aligned} \tag{35}$$

Example-4.1: [Application problem in flood management]

A river basin is prone to flooding, and the water level is modeled by the quadratic function $q(u)=u^2$, where u is the rainfall intensity and $q(u)$ is the water level. The rainfall intensity is expected to be $18mm/h$ for the first hour, $26mm/h$ for the second hour. Using the quadratic functional equation (34), find the maximum water level expected during the flood.

Solution: Let $u_1=18$ and $u_2=26$. Then , we have

$q(u_1)=(18)^2=324;\ q(u_2)=(26)^2=676;\ q(u_1+u_2)=(44)^2=1936,\ q(u_1-u_2)=(-8)^2=64$

$$q\left(\frac{u_1+u_2}{2}\right)=(22)^2=484.$$

Using the quadratic function (30), we can verify that,

LHS of (30)

$$q(u_1)+q\left(\frac{u_1+u_2}{2}\right)=(18)^2+(22)^2=324+484=808.$$

RHS of (30)

$$\left(1+\frac{1}{2^2}\right)q(u_1)+\frac{1}{2^2}q(u_2)\frac{1}{2}\left[\left(\frac{1}{2^2}\right)\left(q\left(u_1+u_2\right)-q\left(u_1-u_2\right)\right)\right]$$
$$=\left(1+\frac{1}{4}\right)(324)+\frac{1}{4}(676)+\frac{1}{2}\left[\frac{1}{4}[1936-64]\right]$$
$$=(324)+\frac{1}{4}(324+676+936)$$
$$=324+484=808.$$

Therefore, the maximum water level expected during the flood is approximately 808 meters.

Example-4.2: **[Application problem in mount sliding impact]**

A mountainous region is prone to landslides, and the impact of a landslide on the surrounding terrain is modeled by the quadratic function $q(u)=u^2$, where u is the volume of the landslide $q(u)$ is the impact. The region is expected to experience three landslides with volumes of 10,000 cubic meters, 20,000 cubic meters and 30,000 cubic meters. Using the quadratic functional equation (35), find the total impact of the three landslides.

Solution: Take $u_1=10{,}000$ $u_2=20{,}000$ and $u_3=30{,}000$.

Using the quadratic function (30), the total impact of the three landslides is approximately 1225,000,000 units.

Example-4.3: [Application problem in pregnancy process]

A healthcare provider is monitoring the growth of a fetus during pregnancy. The growth of the fetus is modeled by the quadratic function $q(u)=u^2$, where u is the number of weeks of pregnancy and $q(u)$ is the fetal growth. The provider wants to estimate the total growth of the fetus over three consecutive trimester, with 10 weeks in the first trimester, 20 weeks in the second trimester, and 30 weeks in the third trimester. Using the quadratic functional equation (35), find the total growth of the fetus.

Solution: Let $u_1=10,$ $u_2=20$ and $u_3=30.$

Using the quadratic function (30), the total growth of the fetus is approximately 1,225 units.

Example-4.4: **[Application problem in satellite launch]**

A satellite launch vehicle has three stages, each with a different fuel capacity. The fuel ca pacity of each stage is modeled by the quadratic function $q(u)=u^2$, where u is the number of fuel tanks and $q(u)$ is the fuel capacity. The first stage has 10 fuel tanks, the second stage has 20 fuel tanks, the third stage has 30 fuel tank, and fourth stage has 40 fuel tanks. Using the quadratic functional equation (1) for $m=4$. Then the total fuel capacity of the launch vehicle is approximately 1,350 units.

Solution: (Hint: Using example-1,2)

Similar way to we can extend many application problems based on the quadratic functional equation (1) in various scenarios on m -variables.

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