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Control Chart Analysis for a Bulk Arrival, Non-Markovian Queue, Random Breakdown, Delay Time, Different Vacation Policies and a Stand by Server with State Dependent Restricted Admissibility of Customers

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Abstract:- The content of the article is a single server bulk arrival non-Markovian queue. The arrival process is a compound Poisson process with arrival size is a random variable follows a discrete distribution. The customers wait in a queue of infinite capacity if the service is not immediate. In queue the first in first out queue discipline is applied. At each service completion point the server apply two different vacation policies. Based on these two models are defined and analysed. The vacation period follows general distribution. The customers are given service singly; the service time follows general distribution. When the server is busy, the server may breakdown, the number of breakdowns follows exponential distribution. After breakdown, there is a potential delay before the repair process starts. The delay period and the repair period are generally distributed. After vacation ends, the server may continue the vacation, called extended vacation follows general distribution. The server is on vacation or repair there is a standby server, the server is service time follows exponential distribution. In addition, the arriving group will be restricted to join the system with state dependent parameter. The model is analysed by obtaining the probability generating function of the number of customers in the queue at various server's states. Numerical illustrations are provided. Control chart analysis is carried out.

Keywords: *Non-Markovian queue- Bulk arrival- Vacation policies- Infinite capacity- FIFO- Extended vacation- Admissibility of customers*

2000 Mathematics Subject Classification Number:*90B22,60K25,60K30*

1. Introduction

Queueing systems are mathematical models used to analyse waiting phenomena in service systems such as communication systems, manufacturing systems, transportations systems and computer systems. In many real valued situations, customers or jobs do not arrive individually but in group or batches and the stochastic processes governing arrivals or services do not satisfy the memoryless property. Such a systems are classified as bulk arrival non-Markovian queues. In a bulk arrival queue, arrivals occur in batches of random size rather than one at a time. The batch size be a discrete random variable, allowing for variability in the number of customers arriving simultaneously. Such a models cannot be analysed using simple birth-death process and instead require advanced techniques such as renewal theory, embedded Markov chain, supplementary variables, probability generating function and transform methods.

Queue with restricted admissibility is a queueing system where arrivals are selectively allowed or denied entries based on capacity, state or policy. That is the arriving customers are admitted only if certain conditions are satisfied. If these conditions are not met, the arriving customers is rejected, blocked or lost without receiving service. The admission to the system may be restricted based on; system capacity, current number of customers in the system, priority rules and probabilistic admission policies. In queues with restricted admissibility help in: Reducing congestion and improving service efficiency.

In a queueing system while working the server may takes break for a random period of time called vacation period. Difference vacation policies are defined by the researchers based on vacation starting point. At each service completion point the server takes two vacation policies. One, the server takes vacation with probability $p(0 < p < 1)$(Policy I). The other the server takes vacation compulsorily (Policy II). Instead of breaking sometimes the server may fails called server breakdown. Once the breakdown occurs the server may send for repair immediately or the server may send for repair after a random period of time. This random period is called delay time. In addition, in some cases if the main server is on vacation or breakdown, the server may be replaced by a stand by server. In this paper the basic model is a non-Markovian bulk arrival queue with vacation, breakdown, delay time, extended vacation, stand by server and restricted admissibility of customer.

2 Review of literature

Many researchers are working on bulk queues. Some notable works in this area are Armero and Conesa (2000), Arumuganathan and Ramaswami (2005), Chang et al., (2004), Fakinos (1991), Srinivasan et al., (2002), Sumita and Masuda (1997), Ushakumari and Krishnamoorthy (1998), Stadje (1989) Ramaswami (1980), Lucantoni (1991) and many others.

The other topic in queueing theory, which has immediate practical application is queue with breakdown, also called queue with unreliable server. Many researchers have contributed on queue with unreliable servers and some noteworthy works are Ke (2005) and Wang (1995,1997).

When a system suddenly stops functioning due to a failure, most of the works available in the literature assume that the repair process on the system starts immediately. However, Khalaf et al (2012) analysed a batch arrival breakdown and delay time. The concept of extended vacation and delay in starting the repair process, some authors introduced the idea of a stand by server in some of the system Madan (1995), Neuts (1984) considered an M/G/1 queue with restriction on the number of customers to be admitted during a service period or with restriction on the time period at which the customers are admitted. Madan and Dayyeh (2002) investigated a bulk queue with restricted admissibility of batches and with Bernoulli schedule server vacation.

3 The Model

A non-Markovian queue has been considered. There is a single server in the queue. At each arrival epochs X number of customers arrive, where X is a random variable whose probability distribution $C_j = Pr\{X = j\}$. The arrival process is compound Poisson process with parameter λ. The arriving customers stays in a waiting line of infinite capacity, if the service is not immediate. The customers are served singly and the service time is generally distributed with distribution function $B(x)$. In the waiting line first in first out queue discipline is applied. At each service completion epoch, the server takes vacation of random length, follows general distribution with distribution function $G_1(x)$. The vacation policies as defined in the introduction part are applied and two models are defined and analyzed. At any instant of time the server may breakdown. The number of breakdowns follows Poisson process with parameter α. After breakdown, there is a potential delay before the repair process starts. The delay period follows general distribution with distribution function $D(x)$. After completion of delay period the repair process starts, the repair period is a random variable with distribution function $R(x)$. At each service completion period, the server takes vacation. There are two policies

Policy I: The server takes vacation with probability $p(0 < p < 1)$

Policy II: The server takes vacation compulsorily.

Based on these policies two models are defined and analyzed. This vacation period is generally distributed with distribution function $G_1(x)$. After vacation period is completed, the server may continue the vacation with probability r. This vacation period is called extended vacation follows general distribution with distribution function $G_2(x)$. When the server is on vacation or repair, there is a standby server, this standby server's service period follows negative exponential distribution with rate μ_2 . In addition, When the server is idle the arriving group will be allowed to join the system with probability p_0 when the server is busy the arriving group will be allowed to join the system with probability p_1 . When the server is on vacation the arriving group will be allowed to join the system with probability p_2 .When the server is repairing the arriving group will be allowed to join the system with probability p_3 . When the server is on extended vacation the arriving group will be allowed to join the system with probability p_4 When the server is in delay period the arriving group will be allowed to join the system with probability p_5 .

The mathematical model is defined using the notations given in the following table 2.1.

Table 2.1: Notations

Notations	Description
λ	Arrival rate
X	Arrival batch service
C_j	$P_r\{X = j\}$
$B(x)$	Service time distribution
$G_1(x)$	Vacation time distribution
α	Breakdown rate
$D(x)$	Delay time distribution
$G_2(x)$	Extended vacation time distribution
$R(x)$	Repair time distribution
γ	Probability of taking extended vacation
μ_2	Stand by server's service rate
$\xi(t)$	The server state at time t
$\mu_1(x)$	Hazard rate function of service time
$\theta_1(x)$	Hazard rate function of vacation time
$\theta_2(x)$	Hazard rate function of extended vacation time
$\gamma(x)$	Hazard rate function of repair time
$\beta(x)$	Hazard rate function of delay time

Since $G_1(x), G_2(x), B(x), D(x)$ and $R(x)$ are cumulative distribution functions, we have

$$G_1(0) = G_2(0) = B(0) = D(0) = R(0) = 0$$
$$G_1(\infty) = G_2(\infty) = B(\infty) = D(\infty) = R(\infty) = 0$$

The Hazard rate function of service time, vacation time, extended vacation time, repair time, delay time are defined as

$$\mu_1(x) = \frac{dB(x)}{1 - B(x)}, \theta_1(x) = \frac{dG_1(x)}{1 - G_1(x)}, \quad \theta_2(x) = \frac{dG_2(x)}{1 - G_2(x)}, \gamma(x) = \frac{dR(x)}{1 - R(x)}, \beta(x) = \frac{dD(x)}{1 - D(x)}$$

We define the following generating functions,

$$P(z; x) = \sum_{n=0}^{\infty} P_n(x) z^n, V(z; x) = \sum_{n=0}^{\infty} V_n(x) z^n, R(z; x) = \sum_{n=0}^{\infty} R_n(x) z^n$$

$$S(z; x) = \sum_{n=0}^{\infty} S_n(x) z^n, U(z; x) = \sum_{n=0}^{\infty} U_n(x) z^n$$

Define $M(t)$ be the number of customers in the queue at time t and the server state $\xi(t)$, the server state at time t.

$$\xi(t) = \begin{cases} 0, & \text{server is idle at time } t \\ 1, & \text{server is busy at time } t \\ 2, & \text{server is on vacation at time } t \\ 3, & \text{server is on extended vacation at time } t \\ 4, & \text{server is in delay at time } t \\ 5, & \text{server is in repair at time } t \end{cases}$$

$\{(M(t), \xi(t)) : t \geq 0\}$ is a bivariate Markov process, where

$\xi = 0, b_1(t), b_2(t), b_3(t), b_4(t), b_5(t)$

The following probabilities are defined for the mathematical analysis

$$Q(t) = P_r\{M(t) = 0; \xi(t) = 0\}$$
$$P_n(x; t) = P_r\{M(t) = n; t \leq b_1(x) \leq t + dt\}$$
$$V_n(x; t) = P_r\{M(t) = n; t \leq b_2(x) \leq t + dt\}$$
$$S_n(x; t) = P_r\{M(t) = n; t \leq b_3(x) \leq t + dt\}$$
$$U_n(x; t) = P_r\{M(t) = n; t \leq b_4(x) \leq t + dt\}$$
$$R_n(x; t) = P_r\{M(t) = n; t \leq b_5(x) \leq t + dt\}$$

The steady state probabilities are as $t \to \infty$

$$P_n(x;t) \to P_n(x);\ V_n(x;t) \to V_n(x);\ S_n(x;t) \to S_n(x);\ U_n(x;t) \to U_n(x); R_n(x;t) \to R_n(x);$$

Based on the arguments in Cox (1955), the following differential-difference equations are obtained:

$$\frac{\partial}{\partial x}P_0(x) = -(p_1\lambda + \mu_1(x) + \alpha)P_0(x) \tag{2.1}$$

$$\frac{\partial}{\partial x}P_n(x) = -(p_1\lambda + \mu_1(x) + \alpha)P_n(x) + p_1\lambda \textstyle\sum_{i=1}^{n} C_i P_{n-i}(x) \quad \text{for } n \geq 1 \tag{2.2}$$

$$\frac{\partial}{\partial x}V_0(x) = -(p_2\lambda + \theta_1(x) + \mu_2)V_0(x) + \mu_2\ V_1(x) \tag{2.3}$$

$$\frac{\partial}{\partial x}V_n(x) = -(p_2\lambda + \theta_1(x) + \mu_2)V_n(x) + p_2\lambda \textstyle\sum_{i=1}^{n} C_i V_{n-i}(x) + \mu_2 V_{n+1}(x), \text{ for } n \geq 1 \tag{2.4}$$

$$\frac{\partial}{\partial x}R_0(x) = -(p_3\lambda + \gamma(x) + \mu_2)R_0(x) + \mu_2\ R_1(x) \tag{2.5}$$

$$\frac{\partial}{\partial x}R_n(x) = -(p_3\lambda + \gamma(x) + \mu_2)R_n(x) + p_3\lambda \textstyle\sum_{i=1}^{n} C_i R_{n-i}(x) + \mu_2 R_{n+1}(x), \text{ for } n \geq 1 \tag{2.6}$$

$$\frac{\partial}{\partial x}S_0(x) = -\big(p_4\lambda + \theta_2(x)\big)S_0(x) \tag{2.7}$$

$$\frac{\partial}{\partial x}S_n(x) = -\big(p_4\lambda + \theta_2(x)\big)S_n(x) + p_4\lambda \textstyle\sum_{i=1}^{n} C_i S_{n-i}(x) \quad \text{for } n \geq 1 \tag{2.8}$$

$$\frac{\partial}{\partial x}U_0(x) = 0 \tag{2.9}$$

$$\frac{\partial}{\partial x}U_n(x) = -\big(p_5\lambda + \beta(x)\big)U_n(x) + p_5\lambda \textstyle\sum_{i=1}^{n} C_i U_{n-i}(x) \quad \text{for } n \geq 1 \tag{2.10}$$

$$p_0\lambda Q = \int_0^\infty R_0(x)\gamma(x)dx + (1-p)\int_0^\infty P_0(x)\mu_1(x) + (1-r)\int_0^\infty V_0(x)\theta_1(x)dx + \int_0^\theta S_0(x)\theta_2(x)dx \tag{2.11}$$

$$p_0\lambda Q = \int_0^\infty R_0(x)\gamma(x)dx + (1-r)\int_0^\infty V_0(x)\theta_1(x)dx + \int_0^\theta S_0(x)\theta_2(x)dx \tag{2.11a}$$

The Boundary conditions are

$$P_n(0) = (1-p)\int_0^\infty P_{n+1}(x)\mu_1(x)dx + (1-r)\int_0^\infty V_{n+1}(x)\theta_1(x)dx + \int_0^\infty S_{n+1}(x)\theta_2(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + p_0\lambda C_{n+1}Q, \text{ for } n \geq 0 \tag{2.12}$$

$$P_n(0) = (1-r)\int_0^\infty V_{n+1}(x)\theta_1(x)dx + \int_0^\infty S_{n+1}(x)\theta_2(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + p_0\lambda C_{n+1}Q\ , \text{ for } n \geq 0 \tag{2.12a}$$

$$V_n(0) = p\int_0^\infty P_n(x)\mu_1(x)dx, \text{ for } n \geq 0 \tag{2.13}$$

$$V_n(0) = \int_0^\infty P_n\,(x)\mu_1(x)dx, \text{ for } n \geq 0 \tag{2.13a}$$

$$S_n(0) = r\int_0^\infty V_n\,(x)\theta_1(x)dx, \text{ for } n \geq 0 \tag{2.14}$$

$$U_n(0) = \int_0^\infty P_{n-1}(x)\alpha dx = \alpha P_{n-1}, \text{ for } n \geq 1 \tag{2.15}$$

$$R_n(0) = \int_0^\infty U_n\,(x)\beta(x)dx, \text{ for } n \geq 1 \tag{2.16}$$

$$R_n(0) = U_n(0) = 0 \tag{2.17}$$

Normalization condition is

$$\sum_{n=0}^{\infty}\int_0^\infty[\,P_n(x)dx + V_n(x)dx + R_n(x)dx + S_n(x)dx + U_n(x)dx] = 1 \tag{2.18}$$

The equations (2.1)- (2.11), (2.12), (2.13)- (2.18) represents the model with vacation policy I.

The equations (2.1)- (2.10), (2.11a), (2.12a), (2.13a), (2.14)- (2.18) represents the model with vacation policy II.

Multiplying equation (2.2) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty}P_n\,(x)z^n = -(p_1\lambda + \mu_1(x) + \alpha)\sum_{n=1}^{\infty}P_n(x)z^n + p_1\lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iP_{n-i}\,(x)z^n$$

Adding the above equation with equation (2.1), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}P_n\,(x)z^n = -(p_1\lambda + \mu_1(x) + \alpha)\sum_{n=0}^{\infty}P_n(x)z^n + p_1\lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iP_{n-i}\,(x)z^n$$

$$\frac{\partial}{\partial x}P(z;x) = [-(p_1\lambda + \mu_1(x) + \alpha) + p_1\lambda C(z)]P(z;x)$$

Where $\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iP_{n-i}\,(x)z^n = C(z)P(z;x)$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$P(z;x) = P(z;0)e^{[-(p_1\lambda+\alpha)+p_1\lambda C(z)]x-\int_0^x\mu_1(u)du} \tag{2.19}$$

Multiplying equation (2.4) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty}V_n\,(x)z^n = -(p_2\lambda + \theta_1(x) + \mu_2)\sum_{n=1}^{\infty}V_n(x)z^n + p_2\lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iV_{n-i}\,(x)z^n + \mu_2\sum_{n=1}^{\infty}V_{n+1}(x)z^n$$

Adding the above equation with equation (2.3), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}V_n\,(x)z^n = -(p_2\lambda + \theta_1(x) + \mu_2)\sum_{n=0}^{\infty}V_n(x)z^n + p_2\lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iV_{n-i}\,(x)z^n + \mu_2\sum_{n=0}^{\infty}V_{n+1}(x)z^n$$

$$\frac{\partial}{\partial x}V(z;x) = \left[-\left(p_2\lambda + \mu_2 + \theta_1(x)\right) + p_2\lambda C(z) + \frac{\mu_2}{z}\right]V(z;x)$$

Where $\sum_{n=1}^{\infty}\sum_{i=1}^{n}C_iV_{n-i}\,(x)z^n = C(z)V(z;x)$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$V(z;x) = V(z;0)e^{\left[-(p_2\lambda+\mu_2)+p_2\lambda C(z)+\frac{\mu_2}{z}\right]x-\int_0^x\theta_1(u)du} \tag{2.20}$$

Multiplying equation (2.6) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty}R_n\,(x)z^n = -(p_3\lambda + \gamma(x) + \mu_2)\sum_{n=1}^{\infty}R_n(x)z^n +$$

$$p_3\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i R_{n-i}(x)z^n + \mu_2 \sum_{n=1}^{\infty} R_{n+1}(x)z^n$$

Adding the above equation with equation (2.5), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} R_n(x)z^n = -(p_3\lambda + \gamma(x) + \mu_2)\sum_{n=0}^{\infty} R_n(x)z^n + p_3\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i R_{n-i}(x)z^n + \mu_2 \sum_{n=0}^{\infty} R_{n+1}(x)z^n$$

$$\frac{\partial}{\partial x}R(z;x) = \left[-\left(p_3\lambda + \mu_2 + \gamma(x)\right) + p_3\lambda C(z) + \frac{\mu_2}{z}\right]R(z;x)$$

Where $\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i R_{n-i}(x)z^n = C(z)R(z;x)$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$R(z;x) = R(z;0)e^{\left[-(p_3\lambda+\mu_2)+p_3\lambda C(z)+\frac{\mu_2}{z}\right]x-\int_0^x \gamma(u)du} \tag{2.21}$$

Multiplying equation (2.8) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} S_n(x)z^n = -\left(p_4\lambda + \theta_2(x)\right)\sum_{n=1}^{\infty} S_n(x)z^n + p_4\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i S_{n-i}(x)z^n$$

Adding the above equation with equation (2.7), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} S_n(x)z^n = -\left(p_4\lambda + \theta_2(x)\right)\sum_{n=0}^{\infty} S_n(x)z^n + p_4\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i S_{n-i}(x)z^n$$

$$\frac{\partial}{\partial x}S(z;x) = [-(p_4\lambda + \theta_2(x)) + p_4\lambda C(z)]S(z;x)$$

Where $\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i S_{n-i}(x)z^n = C(z)S(z;x)$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$S(z;x) = S(z;0)e^{[-p_4\lambda+p_4\lambda C(z)]x-\int_0^x \theta_2(u)du} \tag{2.22}$$

Multiplying equation (2.10) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} U_n(x)z^n = -\left(p_5\lambda + \beta(x)\right)\sum_{n=1}^{\infty} U_n(x)z^n + p_5\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i U_{n-i}(x)z^n$$

Adding the above equation with equation (2.9), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} U_n(x)z^n = -\left(p_5\lambda + \beta(x)\right)\sum_{n=0}^{\infty} U_n(x)z^n + p_5\lambda \sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i U_{n-i}(x)z^n$$

$$\frac{\partial}{\partial x}U(z;x) = [-(p_5\lambda + \beta(x)) + p_5\lambda C(z)]U(z;x)$$

Where $\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i U_{n-i}(x)z^n = C(z)U(z;x)$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$U(z;x) = U(z;0)e^{[-p_5\lambda+p_5\lambda C(z)]x-\int_0^x \beta(u)du} \tag{2.23}$$

Multiplying equation (2.12) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$\begin{aligned} P(z;0) = {} & \frac{1-p}{z}\int_0^{\infty} P(z;x)\mu_1(x)dx + \frac{1-r}{z}\int_0^{\infty} V(z;x)\theta_1(x)dx \\ & + \frac{1}{z}\int_0^{\infty} S(z;x)\theta_2(x)dx + \frac{1}{z}\int_0^{\infty} R(z;x)\gamma(x)dx + \frac{p_0\lambda}{z}c(z)Q \\ & - \frac{1-p}{z}\int_0^{\infty} P_0(x)\mu_1(x)dx \quad - \frac{1-r}{z}\int_0^{\infty} V_0(x)\theta_1(x)dx \end{aligned}$$

$$-\tfrac{1}{z}\textstyle\int_0^\infty S_0(x)\theta_2(x)dx-\tfrac{1}{z}\int_0^\infty R_0(x)\gamma(x)dx \qquad (2.24)$$

Multiplying equation (2.12a) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$P(z;0)=\frac{1-r}{z}\int_0^\infty V(z;x)\theta_1(x)dx+\frac{1}{z}\int_0^\infty S(z;x)\theta_2(x)dx+\frac{1}{z}\int_0^\infty R(z;x)\gamma(x)dx$$
$$+\frac{p_0\lambda}{z}c(z)Q-\frac{1-r}{z}\int_0^\infty V_0(x)\theta_1(x)dx-\frac{1}{z}\int_0^\infty S_0(x)\theta_2(x)dx$$
$$-\tfrac{1}{z}\textstyle\int_0^\infty R_0(x)\gamma(x)dx \qquad (2.24a)$$

Substituting the value of equation (2.11) in (2.24), we have

$$P(z;0)=\frac{1-p}{z}\int_0^\infty P(z;x)\mu_1(x)dx+\frac{1-r}{z}\int_0^\infty V(z;x)\theta_1(x)dx+\frac{1}{z}\int_0^\infty \theta_2(x)S(z;x)dx$$
$$+\tfrac{1}{z}\textstyle\int_0^\infty R(z;x)\gamma(x)dx+\tfrac{p_0\lambda Q}{z}(c(z)-1) \qquad (2.25)$$

Substituting the value of equation (2.11a) in (2.24a), we have

$$P(z;0)=\tfrac{1-r}{z}\textstyle\int_0^\infty V(z;x)\theta_1(x)dx+\tfrac{1}{z}\int_0^\infty \theta_2(x)S(z;x)dx$$
$$+\tfrac{1}{z}\textstyle\int_0^\infty R(z;x)\gamma(x)dx+\tfrac{p_0\lambda Q}{z}(c(z)-1) \qquad (2.25a)$$

Multiplying equation (2.13) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$V(z;0)=p\textstyle\int_0^\infty \mu_1(x)P(z;x)dx \qquad (2.26)$$

Multiplying equation (2.13a) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$V(z;0)=\textstyle\int_0^\infty \mu_1(x)P(z;x)dx \qquad (2.26a)$$

Multiplying equation (2.14) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$S(z;0)=r\textstyle\int_0^\infty \theta_1(x)v(z;x)dx \qquad (2.27)$$

Multiplying equation (2.15) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$U(z;0)=\alpha(z)P(z) \qquad (2.28)$$

Multiplying equation (2.16) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$R(z;0)=\textstyle\int_0^\infty \gamma(x)U(z;x)dx \qquad (2.29)$$

Integrating equations (2.19) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$P(z;x)dx=P(z;0)e^{-Lx-\int_0^x \mu_1(u)du} \qquad (2.30)$$

where, $L=[-(P_1\lambda+\alpha)+P_1\lambda C(z)]$

Multiplying equation (2.30) by $\mu_1(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\textstyle\int_0^\infty P(z;x)\mu_1(x)dx=P(z;0)B^*(L) \qquad (2.31)$$

Integrating equation (2.30) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$P(z)=\textstyle\int_0^\infty P(z;x)dx=P(z;0)\tfrac{[1-B^*(L)]}{L} \qquad (2.32)$$

Integrating equation (2.20) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$V(z;x)dx = V(z;0)e^{-Ax-\int_0^x \theta_1(u)du} \tag{2.33}$$

where, $A = [-(p_2\lambda + \mu_2) + p_2\lambda C(z) + \frac{\mu_2}{z}]$

Multiplying equation (2.33) by $\theta_1(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty V(z;x)\theta_1(x)dx = V(z;0)G_1^*(A) \tag{2.34}$$

Integrating equation (2.33) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$V(z) = \int_0^\infty V(z;x)dx = V(z;0)\frac{[1-G_1^*(A)]}{A} \tag{2.35}$$

Integrating equation (2.21) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$R(z;x)dx = R(z;0)e^{-Cx-\int_0^x \gamma(u)du} \tag{2.36}$$

where, $C = [-(p_3\lambda + \mu_2) + p_3\lambda C(z) + \frac{\mu_2}{z}]$

Multiplying equation (2.36) by $\gamma(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty R(z;x)\gamma(x)dx = R(z;0)R^*(C) \tag{2.37}$$

Integrating equation (2.36) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$R(z) = \int_0^\infty R(z;x)dx = R(z;0)\frac{[1-R^*(C)]}{C} \tag{2.38}$$

Integrating equation (2.22) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$S(z;x)dx = S(z;0)e^{-Hx-\int_0^x \theta_2(u)du} \tag{2.39}$$

where, $H = [-p_4\lambda + p_4\lambda C(z)]$

Multiplying equation (2.39) by $\theta_2(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty S(z;x)\theta_2(x)dx = S(z;0)G_2^*(H) \tag{2.40}$$

Integrating equation (2.39) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$S(z) = \int_0^\infty S(z;x)dx = S(z;0)\frac{[1-G_2^*(H)]}{H} \tag{2.41}$$

Integrating equation (2.23) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$U(z;x)dx = U(z;0)e^{-Ex-\int_0^x \beta(u)du} \tag{2.42}$$

where, $E = [-p_5\lambda + p_5\lambda C(z)]$

Multiplying equation (2.42) by $\beta(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty U(z;x)\beta(x)dx = U(z;0)D^*(E) \tag{2.43}$$

Integrating equation (2.42) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$\int_0^\infty U(z;x)dx = U(z;0)\frac{[1-D^*(E)]}{E} \tag{2.44}$$

Substituting the values of equations (2.31), (2.34), (2.37), (2.40) in (2.25), we have

$$P(z;0) = \frac{1-p}{z}P(z;0)B^*(L) + \frac{1-r}{z}V(z;0)G_1^*(A) + \frac{1}{z}S(z;0)G_2^*(H) + \frac{1}{z}R(z;0)R^*(C) + \frac{p_0\lambda Q}{z}(c(z)-1) \tag{2.45}$$

Substituting the values of equations (2.34), (2.37), (2.40) in (2.25a), we have

$$P(z;0) = \frac{1-r}{z}V(z;0)G_1^*(A) + \frac{1}{z}S(z;0)G_2^*(H) + \frac{1}{z}R(z;0)R^*(C) + \frac{p_0\lambda Q}{z}(c(z)-1) \tag{2.45a}$$

Substituting the value of equation (2.31) in (2.13), we have

$$V(z;0) = pP(z;0)B^*(L) \tag{2.46}$$

Substituting the value of equation (2.31) in (2.13a), we have

$$V(z;0) = P(z;0)B^*(L) \tag{2.46a}$$

Substituting the value of equation (2.34) in (2.14), we have

$$S(z;0) = rV(z;0)G_1^*(A) \tag{2.47}$$

Substituting the value of equation (2.46) in (2.47), we have

$$S(z;0) = rpP(z;0)B^*(L)G_1^*(A) \tag{2.48}$$

Substituting the value of equation (2.46a) in (2.47), we have

$$S(z;0) = rP(z;0)B^*(L)G_1^*(A) \tag{2.48a}$$

Substituting the value of equation (2.32) in (2.15), we have

$$U(z;0) = \alpha zP(z;0)\frac{[1-B^*(L)]}{L} \tag{2.49}$$

Substituting the value of equation (2.43) in (2.16), we have

$$R(z;0) = U(z;0)D^*(E) \tag{2.50}$$

Substituting the value of equation (2.49) in (2.50), we have

$$R(z;0) = \alpha zP(z;0)\frac{[1-B^*(L)]}{L}D^*(E) \tag{2.51}$$

Substituting the value of equation (2.46) , (2.48), (2.51) in (2.45), we have

$$P(z;0) = \frac{Lp_0\lambda Q(c(z)-1)}{J_1} \tag{2.52}$$

where,

$$J_1 = [zL - L(1-p)B^*(L) - L(1-r)pB^*(L)G_1^*(A) - LrpB^*(L)G_1^*(A)G_2^*(H) - \alpha z[1-B^*(L)]D^*(E)R^*(C)$$

Substituting the value of equation (2.46a) , (2.48a), (2.51) in (2.45a), we have

$$P(z;0)=\frac{Lp_0\lambda Q(c(z)-1)}{J_2} \qquad (2.52a)$$

where,

$J_2=[zL-L(1-r)pB^*(L)G_1^*(A)-LrB^*(L)G_1^*(A)G_2^*(H)-\alpha z[1-B^*(L)]D^*(E)R^*(C)$

Now we have to find $H_1(z)$ by adding equations (2.32), (2.35), (2.38), (2.41), (2.44), we have

$$H_1(z)\ =P(z)+V(z)+R(z)+S(z)+U(z)$$

$$H_1(z)\ =P(z;0)\frac{K_1}{LACHE} \qquad (2.53)$$

where, $K_1=ACHE[1-B^*(L)]+LCHEpB^*(L)[1-G_1^*(A)]+AHE\alpha Z[1-B^*(L)]D^*(E)[1-R^*(C)]+\ rpLACEB^*(L)G_1^*(A)[1-G_2^*(H)]+\alpha ZACH[1-B^*(L)][1-D^*(E)]$

Now we have to find $H_2(z)$ by adding equations (2.32), (2.35), (2.38), (2.41), (2.44), we have

$$H_2(z)\ =P(z)+V(z)+R(z)+S(z)+U(z)$$

$$H_2(z)\ =P(z;0)\frac{K_2}{LACHE} \qquad (2.53a)$$

where, $K_2=ACHE[1-B^*(L)]+LCHEB^*(L)[1-G_1^*(A)]+AHE\alpha Z[1-B^*(L)]D^*(E)[1-R^*(C)]+\ rLACEB^*(L)G_1^*(A)[1-G_2^*(H)]+\alpha ZACH[1-B^*(L)][1-D^*(E)]$

Substituting the value of equation (2.52) in (2.53) ,we have

$$H_1(z)=\frac{p_0\lambda Q(C(z)-1)K_1}{ACHEJ_1}\ \text{(Vacation policy I)} \qquad (2.54)$$

where, $n_1=p_0\lambda Q(C(z)-1);n_2=K_1$

$d_1=A;d_2=C;d_3=H;d_4=E;d_5=J_1$

Substituting the value of equation (2.52a) in (2.53a) ,we have

$$H_2(z)=\frac{p_0\lambda Q(C(z)-1)K_2}{ACHEJ_2}\ \text{(Vacation policy II)} \qquad (2.54a)$$

where, $n_1=p_0\lambda Q(C(z)-1);n_2=K_2$

$d_1=A;d_2=C;d_3=H;d_4=E;d_5=J_2$

4. Some performance measures

The following performance measures are derived for the model discussed in section 2

1. Idle probability for vacation policy I

$$Q=\frac{i_1}{i_2} \qquad (3.1)$$

Where $i_1=720q^2\{d_1{}'d_2{}'d_3{}'d_4{}'d_5{}'\}^2$

$$i_2=i_1+5p_0\lambda(2-2q){n_2}^{IV}d_1{}'d_2{}'d_3{}'d_4{}'d_5{}'+2p_0\lambda q{n_2}^{V}d_1{}'d_2{}'d_3{}'d_4{}'d_5{}'-5p_0\lambda{n_2}^{IV}d_1{}''d_2{}'d_3{}'d_4{}'d_5{}'-5p_0\lambda{n_2}^{IV}d_1{}'d_2{}''d_3{}'d_4{}'d_5{}'-$$

$$5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}''{d_4}'{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}'{d_4}''{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}'{d_4}'{d_5}''$$

2. Idle probability for vacation policy II

$$Q = \frac{i_1}{i_2} \tag{3.1a}$$

Where $i_1 = 720q^2\{{d_1}'{d_2}'{d_3}'{d_4}'{d_5}'\}^2$

$$i_2 = i_1 + 5p_0\lambda(2-2q){n_2}^{IV} {d_1}'{d_2}'{d_3}'{d_4}'{d_5}' + 2p_0\lambda q{n_2}^{V} {d_1}'{d_2}'{d_3}'{d_4}'{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}''{d_2}'{d_3}'{d_4}'{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}''{d_3}'{d_4}'{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}''{d_4}'{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}'{d_4}''{d_5}' - 5p_0\lambda {n_2}^{IV} {d_1}'{d_2}'{d_3}'{d_4}'{d_5}''$$

3. Mean number of customers in the queue for vacation policy I

$$M = H'(z) = H'(1) = \frac{D^V N^{VI} - N^V D^{VI}}{6(D^V)^2} \tag{3.2}$$

Where, $D^V = 120{d_1}'{d_2}'{d_3}'{d_4}'{d_5}'$

$$D^{VI} = [360{d_1}''{d_2}'{d_3}'{d_4}'{d_5}' + 360{d_1}'{d_2}''{d_3}'{d_4}'{d_5}' + 360{d_1}'{d_2}'{d_3}''{d_4}'{d_5}' + 360{d_1}'{d_2}'{d_3}'{d_4}''{d_5}' + 360{d_1}'{d_2}'{d_3}'{d_4}'{d_5}'']$$

$$N^V = 5{n_1}'{n_2}^{IV}$$

$$N^{VI} = [15{n_1}''{n_2}^{IV} + 6{n_1}'{n_2}^{V}]$$

Where, ${n_1}' = p_0 Q\lambda E[X]$.

$${n_1}'' = p_0 Q[\lambda E[X^2] - \lambda E[X]],$$

$$d_1' = p_2\lambda E[X] + \mu_2$$

$$d_2' = p_3\lambda E[X] + \mu_2,$$

$$d_3' = p_4\lambda E[X],$$

$$d_4' = p_5\lambda E[X],$$

$${d_1}'' = p_2\lambda[E[X^2] - E[X]],$$

$${d_2}'' = p_3\lambda[E[X^2] - E[X]],$$

$${d_3}'' = p_4\lambda[E[X^2] - E[X]],$$

$${d_4}'' = p_5\lambda[E[X^2] - E[X]],$$

$$d_5' = p_1\lambda E[X][1 - B^*(\alpha)] + \alpha p B^*(\alpha)E[V]d_1' + \alpha r p B^*(\alpha)E[S]d_3' + \alpha E[U]d_4' [1 - B^*(\alpha)] + \alpha E[R]\, d_2'[1 - B^*(\alpha)] + \alpha B^*(\alpha),$$

$${d_5}'' = 2p_1\lambda E[X] + q_1\lambda[E[X^2] - E[X]] - p_1\lambda[E[X^2] - E[X]]B^*(\alpha) -$$

$$\begin{aligned}
&2p_1\lambda E[X]B'^*(\alpha)p_1\lambda E[X]+2p_1\lambda E[X]\,pB^*(\alpha)E[V]d_1'+\;2\alpha p\\
&B'^*(\alpha)p_1\lambda E[X]E[V]d_1'-\alpha pB^*(\alpha)E[V^2][d_1{}']^2+\alpha pB^*(\alpha)\\
&E[V]d_1''+2p_1\lambda E[X]rpB^*(\alpha)E[S]d_3'+\alpha rpB'^*(\alpha)E[S]d_3'\\
&p_1\lambda E[X]-2\alpha rpB^*(\alpha)E[V]E[S]d_3'd_1'-\alpha rpB^*(\alpha)E[S^2][d_3{}']^2\\
&+\alpha rpB^*(\alpha)E[S]d_3''+\;2\alpha E[U]d_3'+2\alpha E[R]\,d_2'\;-\alpha E[U^2][d_4{}']^2\\
&+\alpha E[U]d_4''-2\alpha E[U]d_4'E[R]\,d_2'-\alpha\,E[R^2][d_2{}']^2+2\alpha B'^*(\alpha)\\
&p_1\lambda E[X]-2\alpha d_4'B^*(\alpha)E[U]-2\alpha B^*(\alpha)E[R]\,d_2'-2\alpha B'^*(\alpha)\\
&p_1\lambda E[X]E[X]E[U]d_4'-2\alpha B'^*(\alpha)[q_1\lambda E[X]+\gamma_1]E[R]\,d_2'+\alpha B^*(\alpha)\\
&E[U^2][d_4{}']^2-\alpha B^*(\alpha)E[U]\,d_4''+2\alpha B^*(\alpha)E[U]\,d_4'E[R]\,d_2'+\alpha\\
&B^*(\alpha)E[R^2][d_2{}']^2-\alpha B^*(\alpha)E[R]\,d_2{}''
\end{aligned}$$

$$\begin{aligned}
n_2{}^{IV}=\;&24d_1{}'d_2{}'d_3{}'d_4{}'[1-B^*(\alpha)]+\;\alpha pB^*(\alpha)E[V]+\alpha[1-B^*(\alpha)]E[R]+\alpha rp\\
&B^*(\alpha)E[S]+\alpha[1-B^*(\alpha)]E[U]
\end{aligned}$$

$$\begin{aligned}
n_2{}^{V}=\;&42d_1{}''d_2{}'d_3{}'d_4{}'\,[1-B^*(\alpha)]+42d_1{}'d_2{}''d_3{}'d_4{}'\,[1-B^*(\alpha)]+42d_1{}'d_2{}'d_3{}''d_4{}'\\
&[1-B^*(\alpha)]+42d_1{}'d_2{}'d_3{}'d_4{}''[1-B^*(\alpha)]-96d_1{}'d_2{}'d_3{}'d_4{}'B'^*(\alpha)p_1\lambda E[X]+\\
&96p_1\lambda E[X]d_2{}'d_3{}'d_4{}'pB^*(\alpha)E[V]d_1'+42\alpha d_2{}''d_3{}'d_4{}'pB^*(\alpha)E[V]d_1'+\\
&42\alpha d_2{}'d_3{}''\,d_4{}'pB^*(\alpha)E[V]d_1'+42\alpha d_2'd_3{}'d_4{}''pB^*(\alpha)E[V]d_1'+\\
&96\alpha d_2'd_3{}'d_4{}'pB'^*(\alpha)\,p_1\lambda E[X]E[V]d_1'-42\alpha d_2{}'d_3{}'d_4{}'pB^*(\alpha)E[V^2][d_1{}']^2+\\
&42\alpha d_2{}'d_3{}'d_4{}'pB^*(\alpha)E[V]d_1'+42\alpha d_1{}''d_3{}'d_4{}'[1-B^*(\alpha)]E[R]\,d_2'+\\
&42\alpha d_1{}'d_3{}''d_4{}'\;\;[1-B^*(\alpha)]E[R]\,d_2'+42\alpha d_1{}'d_3{}'d_4{}''[1-B^*(\alpha)]E[R]\,d_2'+\\
&96\alpha d_1{}'d_3{}'d_4{}'[1-B^*(\alpha)]E[R]\,d_2'+96\alpha d_1{}'d_3{}'d_4{}'B'^*(\alpha)p_1\lambda E[X]E[R]d_2'-\\
&96\alpha d_1{}'d_3{}'\,d_2'[1-B^*(\alpha)]E[R]E[U][d_4{}']^2-\;42\alpha d_1{}'d_3{}'d_4{}'[1-\\
&B^*(\alpha)]\,E[R^2][d_2{}']^2+42\alpha d_1{}'d_3{}'d_4{}'E[R]\,[1-B^*(\alpha)]d_2{}''+\\
&96rpp_1\lambda E[X]d_1{}'d_2{}'d_4{}'B^*(\alpha)E[S]d_3'\;+42\alpha rpd_1{}''d_2{}'d_4{}'B^*(\alpha)E[S]d_3'+\\
&42\alpha rpd_1{}'d_2{}''d_4{}'B^*(\alpha)E[S]d_3'+96\alpha rpd_1{}'d_2{}'d_4{}'\,B'^*(\alpha)p_1\lambda E[X]E[S]d_3'-\\
&96\alpha rp[d_1{}']^2d_2{}'d_4{}'B^*(\alpha)E[V]E[S]d_3'\;+42\alpha rpd_1{}'d_2{}'d_4{}''B^*(\alpha)E[S]d_3'\;-\\
&42\alpha rpd_1{}'d_2{}'d_4{}'B^*(\alpha)E[S^2][d_3{}']^2+42\alpha rpd_1{}'d_2{}'d_4{}'B^*(\alpha)E[S]d_3''+\\
&96\alpha d_1{}'d_2{}'d_3{}'[1-B^*(\alpha)]E[U]d_4'+42\alpha d_1{}''d_2{}'d_3{}'E[U][1-B^*(\alpha)]d_4{}'+\\
&42\alpha d_1{}'d_2{}''d_3{}'E[U][1-B^*(\alpha)]d_4{}'+42\alpha d_1{}'d_2{}'d_3{}''E[U][1-B^*(\alpha)]d_4{}'+\\
&96\alpha d_1{}'d_2{}'d_3{}'B'^*(\alpha)p_1\lambda E[X]\,E[U]d_4'-42\alpha d_1{}'d_2{}'d_3{}'[1-\\
&B^*(\alpha)]\,E[U^2][d_4{}']^2+42\alpha d_1{}'d_2{}'d_3{}'[1-B^*(\alpha)]\,E[U]d_4{}''
\end{aligned}$$

4. Mean number of customers in the queue for vacation policy II

$$M=H'(z)=H'(1)=\frac{D^VN^{VI}-N^VD^{VI}}{6(D^V)^2}\qquad(3.2a)$$

Where, $D^V=120d_1{}'d_2{}'d_3{}'d_4{}'d_5{}'$

$$\begin{aligned}
D^{VI}=\;&[360d_1{}''d_2{}'d_3{}'d_4{}'d_5{}'+360d_1{}'d_2{}''d_3{}'d_4{}'d_5{}'+360d_1{}'d_2{}'d_3{}''d_4{}'d_5{}'+\\
&360d_1{}'d_2{}'d_3{}'d_4{}''d_5{}'+360d_1{}'d_2{}'d_3{}'d_4{}'d_5{}'']
\end{aligned}$$

$$N^V=5n_1{}'n_2{}^{IV}$$

$$N^{VI}=[15n_1{}''n_2{}^{IV}+6n_1{}'n_2{}^{V}]$$

Where, $n_1{}'=p_0Q\lambda E[X]$.

$$n_1'' = p_0 Q\left[\lambda E[X^2] - \lambda E[X]\right],$$

$$d_1' = p_2 \lambda E[X] + \mu_2$$

$$d_2' = p_3 \lambda E[X] + \mu_2,$$

$$d_3' = p_4 \lambda E[X],$$

$$d_4' = p_5 \lambda E[X],$$

$$d_1'' = p_2 \lambda\left[E[X^2] - E[X]\right],$$

$$d_2'' = p_3 \lambda\left[E[X^2] - E[X]\right],$$

$$d_3'' = p_4 \lambda\left[E[X^2] - E[X]\right],$$

$$d_4'' = p_5 \lambda\left[E[X^2] - E[X]\right],$$

$$\begin{aligned} d_5' = {} & p_1 \lambda E[X][1 - B^*(\alpha)] + \alpha B^*(\alpha)E[V]d_1' + \alpha r B^*(\alpha)E[S]d_3' \\ & + \alpha E[U]d_4'[1 - B^*(\alpha)] + \alpha E[R]\, d_2'[1 - B^*(\alpha)] + \alpha B^*(\alpha), \end{aligned}$$

$$\begin{aligned} d_5'' = {} & 2p_1\lambda E[X] + p_1\lambda\left[E[X^2] - E[X]\right] - p_1\lambda\left[E[X^2] - E[X]\right]B^*(\alpha) \; - \; 2p_1\lambda E[X] \\ & B'^*(\alpha)p_1\lambda E[X] + 2p_1\lambda E[X]\, B^*(\alpha)E[V]d_1' + \; 2\alpha B'^*(\alpha)p_1\lambda E[X]E[V]d_1' \; - \\ & \alpha B^*(\alpha)E[V^2][d_1{}']^2 \; + \alpha B^*(\alpha)E[V]d_1'' + 2p_1\lambda E[X]rB^*(\alpha)E[S]d_3' + \alpha r B'^*(\alpha) \\ & E[S]d_3' p_1\lambda E[X] - 2\alpha r B^*(\alpha)E[V]E[S]d_3' d_1' - \alpha r B^*(\alpha)E[S^2][d_3{}']^2 \\ & + \alpha r B^*(\alpha)E[S]d_3'' + \; 2\alpha E[U]d_3' + 2\alpha E[R]\, d_2' \; - \alpha\, E[U^2][d_4{}']^2 + \alpha E[U]d_4'' - \\ & 2\alpha E[U]d_4' E[R]\, d_2' - \alpha\, E[R^2][d_2{}']^2 + 2\alpha B'^*(\alpha)p_1\lambda E[X] - 2\alpha d_4' B^*(\alpha)E[U] \\ & -2\alpha B^*(\alpha)E[R]\, d_2' - 2\alpha B'^*(\alpha)p_1\lambda E[X]E[X]E[U]d_4' - 2\alpha B'^*(\alpha)p_1\lambda E[X] \\ & E[R]\, d_2' + \alpha B^*(\alpha)E[U^2][d_4{}']^2 - \alpha B^*(\alpha)E[U]\, d_4'' + 2\alpha B^*(\alpha)E[U]\, d_4' E[R] \\ & d_2' + \alpha B^*(\alpha)E[R^2][d_2{}']^2 - \alpha B^*(\alpha)E[R]\, d_2'' \end{aligned}$$

$$\begin{aligned} n_2{}^{IV} = {} & 24 d_1{}' d_2{}' d_3{}' d_4{}'[1 - B^*(\alpha)] + \; \alpha B^*(\alpha)E[V] + \alpha[1 - B^*(\alpha)]E[R] + \alpha r B^*(\alpha) \\ & E[S] + \alpha[1 - B^*(\alpha)]E[U] \end{aligned}$$

$$\begin{aligned} n_2{}^{V} = {} & 42 d_1{}'' d_2{}' d_3{}' d_4{}'\, [1 - B^*(\alpha)] + 42 d_1{}' d_2{}'' d_3{}' d_4{}'\, [1 - B^*(\alpha)] + 42 d_1{}' d_2{}' d_3{}'' d_4{}' \\ & [1 - B^*(\alpha)] + 42 d_1{}' d_2{}' d_3{}' d_4{}''[1 - B^*(\alpha)] - 96 d_1{}' d_2{}' d_3{}' d_4{}' B'^*(\alpha)p_1\lambda E[X] + \\ & 96 p_1\lambda E[X] d_2{}' d_3{}' d_4{}' B^*(\alpha)E[V]d_1' + 42\alpha d_2{}'' d_3{}' d_4{}' B^*(\alpha)E[V]d_1' + \\ & 42\alpha d_2{}' d_3{}''\, d_4{}' B^*(\alpha)E[V]d_1' + 42\alpha d_2' d_3{}' d_4{}'' B^*(\alpha)E[V]d_1' + \\ & 96\alpha d_2' d_3{}' d_4{}' B'^*(\alpha)\, p_1\lambda E[X]E[V]d_1' - 42\alpha d_2{}' d_3{}' d_4{}' B^*(\alpha)E[V^2][d_1{}']^2 + \\ & 42\alpha d_2{}' d_3{}' d_4{}' B^*(\alpha)E[V]d_1' + 42\alpha d_1{}'' d_3{}' d_4{}'[1 - B^*(\alpha)]E[R]\, d_2' + \\ & 42\alpha d_1{}' d_3{}'' d_4{}'\; [1 - B^*(\alpha)]E[R]\, d_2' + 42\alpha d_1{}' d_3{}' d_4{}''[1 - B^*(\alpha)]E[R]\, d_2' + \\ & 96\alpha d_1{}' d_3{}' d_4{}'[1 - B^*(\alpha)]E[R]\, d_2' + 96\alpha d_1{}' d_3{}' d_4{}' B'^*(\alpha)p_1\lambda E[X]E[R]d_2' - \\ & 96\alpha d_1{}' d_3{}'\, d_2'[1 - B^*(\alpha)]E[R]E[U][d_4{}']^2 - \; 42\alpha d_1{}' d_3{}' d_4{}'[1 - \\ & B^*(\alpha)]\, E[R^2][d_2{}']^2 + 42\alpha d_1{}' d_3{}' d_4{}' E[R]\, [1 - B^*(\alpha)]d_2{}'' + \\ & 96 r p_1\lambda E[X]d_1{}' d_2{}' d_4{}' B^*(\alpha)E[S]d_3' \; + 42\alpha r d_1{}'' d_2{}' d_4{}' B^*(\alpha)E[S]d_3' + \\ & 42\alpha r d_1{}' d_2{}'' d_4{}' B^*(\alpha)E[S]d_3' + 96\alpha r d_1{}' d_2{}' d_4{}'\, B'^*(\alpha)p_1\lambda E[X]E[S]d_3' - \\ & 96\alpha r[d_1{}']^2 d_2{}' d_4{}' B^*(\alpha)E[V]E[S]d_3' \; + 42\alpha r d_1{}' d_2{}' d_4{}'' B^*(\alpha)E[S]d_3' \; - \end{aligned}$$

$$42\alpha r d_1'd_2'd_4'B^*(\alpha)E[S^2][d_3']^2 + 42\alpha r d_1'd_2'd_4'B^*(\alpha)E[S]d_3'' + 96\alpha d_1'd_2'd_3'[1 - B^*(\alpha)]E[U]d_4' + 42\alpha d_1''d_2'd_3'E[U][1 - B^*(\alpha)]d_4' + 42\alpha d_1'd_2''d_3'E[U][1 - B^*(\alpha)]d_4' + 42\alpha d_1'd_2'd_3''E[U][1 - B^*(\alpha)]d_4' + 96\alpha d_1'd_2'd_3'B'^*(\alpha)p_1\lambda E[X]\,E[U]d_4' - 42\alpha d_1'd_2'd_3'[1 - B^*(\alpha)]\,E[U^2][d_4']^2 + 42\alpha d_1'd_2'd_3'[1 - B^*(\alpha)]\,E[U]d_4''$$

5. Variance number of customers in the queue

$$V = H''(1) + H'(1) - [H'(1)]^2 \tag{3.3}$$

$$H''(z) = H''(1) = \frac{91N^V}{1260} \tag{3.4}$$

6. Variance number of customers in the queue

$$V = H''(1) + H'(1) - [H'(1)]^2 \tag{3.3a}$$

$$H''(z) = H''(1) = \frac{91N^V}{1260} \tag{3.4a}$$

7. Numerical Illustrations:

The model analysed in this article is numerically analysed by assuming the general distribution has negative exponential distribution with following suitable parameters.

$$E[X] = \frac{1}{q}, E[V] = \frac{1}{\theta_1} E[S] = \frac{1}{\theta_2} E[R] = \frac{1}{\gamma} E[U] = \frac{1}{\beta}, E[X^2] = \frac{2-q}{q^2},$$

$$E[V^2] = \frac{2-\theta_1}{{\theta_1}^2}, E[S^2] = \frac{2-\theta_2}{{\theta_2}^2}, E[R^2] = \frac{2-\gamma}{\gamma^2}, E[R^2] = \frac{2-\beta}{\beta^2},$$

$$B^*(\alpha) = \frac{\mu_1}{\alpha+\mu_1}, B'^*(\alpha) = \frac{-\mu_1}{(\alpha+\mu_1)^2}$$

The following performance measures in section 3 are calculated

1. Idle probability for vacation policy I

$$Q = \frac{i_1}{i_2} \tag{4.1}$$

Where $i_1 = 720q^2\{d_1'd_2'd_3'd_4'd_5'\}^2$

$$i_2 = a_1 + 5p_0\lambda(2-2q){n_2}'^V d_1'd_2'd_3'd_4'd_5' + 2p_0\lambda q {n_2}^v d_1'd_2'd_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1''d_2'd_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2''d_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3''d_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3'd_4''d_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3'd_4'd_5''$$

2. Idle probability for vacation policy II

$$Q = \frac{i_1}{i_2} \tag{4.1a}$$

Where $i_1 = 720q^2\{d_1'd_2'd_3'd_4'd_5'\}^2$

$$i_2 = a_1 + 5p_0\lambda(2-2q){n_2}'^V d_1'd_2'd_3'd_4'd_5' + 2p_0\lambda q {n_2}^v d_1'd_2'd_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1''d_2'd_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2''d_3'd_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3''d_4'd_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3'd_4''d_5' - 5p_0\lambda {n_2}'^V d_1'd_2'd_3'd_4'd_5''$$

3. Mean number of customers in the queue

$$M = H'(z) = H'(1) = \frac{D^{V}N^{V\prime} - N^{V}D^{V\prime}}{6(D^{V})^{2}} \qquad (4.2)$$

Where, $n_1{}' = \frac{p_0\lambda Q}{q}$

$$n_1{}'' = \frac{p_0\lambda Q[2-2q]}{q^2}$$

$$n_2{}'^{V} = \frac{A_1}{q^4(\alpha+\mu_{1)}\theta_1\theta_2\beta\gamma}$$

$$d_1' = \frac{p_2\lambda+q\mu_2}{q}$$

$$d_2' = \frac{p_3\lambda+q\mu_2}{q}$$

$$d_3' = \frac{p_4\lambda}{q}$$

$$d_4' = \frac{p_5\lambda}{q}$$

$$d_1'' = \frac{p_2\lambda[2-2q]}{q^2}$$

$$d_2'' = \frac{p_3\lambda[2-2q]}{q^2}$$

$$d_3'' = \frac{p_4\lambda[2-2q]}{q^2}$$

$$d_4'' = \frac{p_5\lambda[2-2q]}{q^2}$$

$$d_5{}' = \frac{L_1}{q(\alpha+\mu_1)\theta_1\theta_2\beta\gamma}$$

$$L_1 = \alpha p_1\lambda\theta_1\theta_2\beta\gamma + \alpha p\mu_1[p_2\lambda + q\mu_2]\,\theta_2\beta\gamma + \alpha r p\mu_1 p_4\lambda\theta_1\beta\gamma + \alpha^2 p_5\lambda\theta_1\theta_2\gamma + \alpha^2[p_3\lambda + q\mu_2]\theta_1\theta_2\beta + \alpha\mu_1 q\theta_1\theta_2\beta\gamma$$

$$d_5{}'' = \frac{L_2-L_3}{q^2(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2}$$

$$L_2 = 2p_1\lambda q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + p_1\lambda[2-2q](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2[p_1\lambda]^2\mu_1{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2p\mu_1p_1\lambda[p_2\lambda+q\mu_2]\,(\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + \alpha p\mu_1p_2\lambda[2-2q]\,(\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 2rp\mu_1p_1\lambda p_4\lambda(\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + \alpha r p\mu_1p_4\lambda[2-2q]{\theta_1}^2\theta_2\beta^2\gamma^2(\alpha+\mu_1) + 2\alpha p_4\lambda q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha[p_3\lambda+q\mu_2]q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma + \alpha p_5\lambda[2-2q](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha\mu_1p_1\lambda p_5\lambda{\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha\mu_1p_1\lambda[p_3\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma + \alpha\mu_1[p_5\lambda]^2[2-\beta](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\gamma^2 + \alpha\mu_1[2-\gamma][p_3\lambda+q\mu_2]^2(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2$$

$$L_3 = \mu_1p_1\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2\alpha p\mu_1p_1\lambda[q_2\lambda+q\mu_2]\theta_1{\theta_2}^2\beta^2\gamma^2 + \alpha p\mu_1[2-\theta_1][p_2\lambda+q\mu_2]^2\,(\alpha+\mu_1){\theta_2}^2\beta^2\gamma^2 + \alpha r p\mu_1p_1\lambda p_4\lambda{\theta_1}^2\theta_2\beta^2\gamma^2 +$$

$$\begin{aligned}
&2\alpha rp\mu_1p_4\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1\theta_2\beta^2\gamma^2+\alpha rp\mu_1[2-\theta_2][p_4\lambda]^2(\alpha+\\
&\mu_1){\theta_1}^2\beta^2\gamma^2+\ \alpha[p_5\lambda]^2[2-\beta](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\gamma^2+2\alpha p_5\lambda[p_3\lambda+q\mu_2](\alpha+\\
&\mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma+\alpha[2-\gamma][p_3\lambda+q\mu_2]^2(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2+\\
&2\alpha\mu_1p_1\lambda q{\theta_1}^2{\theta_2}^2\beta^2\gamma^2+2\alpha\mu_1p_5\lambda q(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2+2\alpha\mu_1q(\alpha+\mu_1)[p_3\lambda+\\
&q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma+\alpha\mu_1p_5\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2+\ 2\alpha\mu_1p_5\lambda(\alpha+\mu_1)[p_3\lambda+\\
&q\mu_2]{\theta_1}^2{\theta_2}^2\beta\gamma+\ \alpha\mu_1p_3\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma
\end{aligned}$$

Where,

$$\begin{aligned}
A_1=\ &24[p_2\lambda+q\mu_2]p_4\lambda p_5\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1)\theta_1\theta_2\beta\gamma[\alpha q^4\theta_1\theta_2\beta\gamma+\alpha p\mu_1q^4\theta_2\\
&+\alpha^2q^4\theta_1\theta_2\beta+\alpha rp\mu_1q^4\theta_1\beta\gamma+\alpha^2q^4\theta_1\theta_2\gamma
\end{aligned}$$

$${n_2}^V=\frac{A_2-A_3}{q^5(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2}$$

$$\begin{aligned}
A_2=\ &42\alpha p_2\lambda[2-2q][p_3\lambda+q\mu_2]p_4\lambda p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2+42\alpha p_3\lambda[2-\\
&2q][p_2\lambda+q\mu_2]p_4\lambda p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2+42\alpha[p_2\lambda+q\mu_2][p_3\lambda+\\
&q\mu_2]p_4\lambda[2-2q]p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2+42\alpha[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2]p_5\lambda[2-\\
&2q]p_4\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2+96\mu_1p_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+\\
&q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma^2+96p\mu_1p_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2+\\
&42\alpha p\mu_1p_3\lambda[2-2q]p_4\lambda p_5\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2+42\alpha p\mu_1p_4\lambda[2-\\
&2q][p_3\lambda+q\mu_2]\ p_5\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2+42\alpha p\mu_1[p_3\lambda+\\
&q\mu_2]p_4\lambda p_5\lambda[2-2q][p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2+42\alpha p\mu_1[p_3\lambda+\\
&q\mu_2]p_4\lambda p_5\lambda p_2\lambda[2-2q](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2+42\alpha^2p_2\lambda[2-2q]p_4\lambda p_5\lambda[p_3\lambda+\\
&q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma+42\alpha^2[p_2\lambda+q\mu_2]p_4\lambda[2-2q][p_3\lambda+q\mu_2]p_5\lambda(\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma+42\alpha^2[p_2\lambda+q\mu_2]p_5\lambda[2-2q][p_3\lambda+q\mu_2]p_4\lambda(\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma+96\alpha^2q[p_2\lambda+q\mu_2]p_4\lambda p_5\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma+\\
&42\alpha^2p_4\lambda p_5\lambda p_3\lambda[2-2q][p_2\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma+\\
&96\mu_1rpp_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2+42\alpha rp\mu_1p_2\lambda[2-\\
&2q]p_4\lambda p_5\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2+42\alpha rp\mu_1p_4\lambda p_5\lambda p_3\lambda[2-2q][p_2\lambda+\\
&q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2+42\alpha rp\mu_1p_4\lambda p_5\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+q\mu_2](\alpha+\\
&\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2+42\alpha rp\mu_1p_5\lambda p_4\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+q\mu_2](\alpha+\\
&\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2+96\alpha^2qp_5\lambda p_4\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\beta\gamma^2+\\
&42\alpha^2p_2\lambda[2-2q]p_5\lambda p_4\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2+\ 42\alpha^2p_4\lambda p_5\lambda p_3\lambda[2-\\
&2q][p_2\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\ \beta\gamma^2+42\alpha^2p_4\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+\\
&q\mu_2]p_5\lambda\ (\alpha+\mu_1){\theta_1}^2{\theta_2}^2\ \beta\gamma^2+\ 42\alpha^2p_4\lambda\ [p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]p_5\lambda[2-2q](\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\ \beta\gamma^2
\end{aligned}$$

4. Mean number of customers in the queue

$$M=H'(z)=H'(1)=\frac{D^VN^{V\prime}-N^VD^{V\prime}}{6(D^V)^2} \tag{4.2a}$$

Where, ${n_1}'=\frac{p_0\lambda Q}{q}$

$$n_1{}'' = \frac{p_0\lambda Q[2-2q]}{q^2}$$

$$n_2{}'^{V} = \frac{A_1}{q^4(\alpha+\mu_{1)}\theta_1\theta_2\beta\gamma}$$

$$d_1' = \frac{p_2\lambda+q\mu_2}{q}$$

$$d_2' = \frac{p_3\lambda+q\mu_2}{q}$$

$$d_3' = \frac{p_4\lambda}{q}$$

$$d_4' = \frac{p_5\lambda}{q}$$

$$d_1'' = \frac{p_2\lambda[2-2q]}{q^2}$$

$$d_2'' = \frac{p_3\lambda[2-2q]}{q^2}$$

$$d_3'' = \frac{p_4\lambda[2-2q]}{q^2}$$

$$d_4'' = \frac{p_5\lambda[2-2q]}{q^2}$$

$$d_5{}' = \frac{L_1}{q(\alpha+\mu_1)\theta_1\theta_2\beta\gamma}$$

$$\begin{aligned} L_1 = {} & \alpha p_1\lambda\theta_1\theta_2\beta\gamma + \alpha\mu_1[p_2\lambda + q\mu_2]\,\theta_2\beta\gamma + \alpha r\mu_1 p_4\lambda\theta_1\beta\gamma + \alpha^2 p_5\lambda\theta_1\theta_2\gamma + \\ & \alpha^2[p_3\lambda + q\mu_2]\theta_1\theta_2\beta + \alpha\mu_1 q\theta_1\theta_2\beta\gamma \end{aligned}$$

$$d_5{}'' = \frac{L_2-L_3}{q^2(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2}$$

$$\begin{aligned} L_2 = {} & 2p_1\lambda q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + p_1\lambda[2-2q](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + \\ & 2[p_1\lambda]^2\mu_1{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2\mu_1 p_1\lambda[p_2\lambda+q\mu_2]\,(\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + \\ & \alpha p\mu_1 p_2\lambda[2-2q]\,(\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 2r\mu_1 p_1\lambda p_4\lambda(\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + \\ & \alpha r\mu_1 p_4\lambda[2-2q]{\theta_1}^2\theta_2\beta^2\gamma^2(\alpha+\mu_1) + 2\alpha p_4\lambda q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma^2 + \\ & 2\alpha[p_3\lambda+q\mu_2]q(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma + \alpha p_5\lambda[2-2q](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma^2 + \\ & 2\alpha\mu_1 p_1\lambda p_5\lambda{\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha\mu_1 p_1\lambda[p_3\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma + \alpha\mu_1[p_5\lambda]^2[2-\beta](\alpha + \\ & \mu_1){\theta_1}^2{\theta_2}^2\gamma^2 + \alpha\mu_1[2-\gamma][p_3\lambda+q\mu_2]^2(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2 \end{aligned}$$

$$\begin{aligned} L_3 = {} & \mu_1 p_1\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2\alpha p\mu_1 p_1\lambda[q_2\lambda+q\mu_2]\theta_1{\theta_2}^2\beta^2\gamma^2 + \\ & \alpha\mu_1[2-\theta_1][p_2\lambda+q\mu_2]^2\,(\alpha+\mu_1){\theta_2}^2\beta^2\gamma^2 + \alpha r\mu_1 p_1\lambda p_4\lambda{\theta_1}^2\theta_2\beta^2\gamma^2 + \\ & 2\alpha r\mu_1 p_4\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1\theta_2\beta^2\gamma^2 + \alpha r\mu_1[2-\theta_2][p_4\lambda]^2(\alpha + \\ & \mu_1){\theta_1}^2\beta^2\gamma^2 + \alpha[p_5\lambda]^2[2-\beta](\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\gamma^2 + 2\alpha p_5\lambda[p_3\lambda+q\mu_2](\alpha + \\ & \mu_1)^2{\theta_1}^2{\theta_2}^2\beta\gamma + \alpha[2-\gamma][p_3\lambda+q\mu_2]^2(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2 + \\ & 2\alpha\mu_1 p_1\lambda q{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 2\alpha\mu_1 p_5\lambda q(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha\mu_1 q(\alpha+\mu_1)[p_3\lambda + \end{aligned}$$

$$\begin{aligned}&q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma + \alpha\mu_1 p_5\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2 + 2\alpha\mu_1 p_5\lambda(\alpha+\mu_1)[p_3\lambda +\\ &q\mu_2]{\theta_1}^2{\theta_2}^2\beta\gamma + \alpha\mu_1 p_3\lambda[2-2q](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma\end{aligned}$$

Where, $A_1 = 24[p_2\lambda + q\mu_2]p_4\lambda p_5\lambda[p_3\lambda + q\mu_2](\alpha + \mu_1)\theta_1\theta_2\beta\gamma[\alpha q^4\theta_1\theta_2\beta\gamma + \alpha p\mu_1 q^4\theta_2\beta\gamma + \alpha^2 q^4\theta_1\theta_2\beta + \alpha r\mu_1 q^4\theta_1\beta\gamma + \alpha^2 q^4\theta_1\theta_2\gamma$

$$n_2{}^V = \frac{A_2 - A_3}{q^5(\alpha+\mu_1)^2{\theta_1}^2{\theta_2}^2\beta^2\gamma^2}$$

$$\begin{aligned}A_2 = {}&42\alpha p_2\lambda[2-2q][p_3\lambda+q\mu_2]p_4\lambda p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 42\alpha p_3\lambda[2-\\
&2q][p_2\lambda+q\mu_2]p_4\lambda p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 42\alpha[p_2\lambda+q\mu_2][p_3\lambda+\\
&q\mu_2]p_4\lambda[2-2q]p_5\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 42\alpha[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2]p_5\lambda[2-\\
&2q]p_4\lambda(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 96\mu_1 p_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+\\
&q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma^2 + 96\mu_1 p_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 +\\
&42\alpha\mu_1 p_3\lambda[2-2q]p_4\lambda p_5\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 42\alpha\mu_1 p_4\lambda[2-\\
&2q][p_3\lambda+q\mu_2]\,p_5\lambda[p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 42\alpha\mu_1[p_3\lambda+\\
&q\mu_2]p_4\lambda p_5\lambda[2-2q][p_2\lambda+q\mu_2](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 42\alpha\mu_1[p_3\lambda+\\
&q\mu_2]p_4\lambda p_5\lambda p_2\lambda[2-2q](\alpha+\mu_1)\theta_1{\theta_2}^2\beta^2\gamma^2 + 42\alpha^2 p_2\lambda[2-2q]p_4\lambda p_5\lambda[p_3\lambda+\\
&q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma + 42\alpha^2[p_2\lambda+q\mu_2]p_4\lambda[2-2q][p_3\lambda+q\mu_2]p_5\lambda(\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma + 42\alpha^2[p_2\lambda+q\mu_2]p_5\lambda[2-2q][p_3\lambda+q\mu_2]p_4\lambda(\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma + 96\alpha^2 q[p_2\lambda+q\mu_2]p_4\lambda p_5\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma +\\
&42\alpha^2 p_4\lambda p_5\lambda p_3\lambda[2-2q][p_2\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma +\\
&96\mu_1 r p_1\lambda p_4\lambda p_5\lambda[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + 42\alpha r\mu_1 p_2\lambda[2-\\
&2q]p_4\lambda p_5\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + 42\alpha r\mu_1 p_4\lambda p_5\lambda p_3\lambda[2-2q][p_2\lambda+\\
&q\mu_2](\alpha+\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + 42\alpha r\mu_1 p_4\lambda p_5\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+q\mu_2](\alpha+\\
&\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + 42\alpha r\mu_1 p_5\lambda p_4\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+q\mu_2](\alpha+\\
&\mu_1){\theta_1}^2\theta_2\beta^2\gamma^2 + 96\alpha^2 q p_5\lambda p_4\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\beta\gamma^2 +\\
&42\alpha^2 p_2\lambda[2-2q]p_5\lambda p_4\lambda[p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta\gamma^2 + 42\alpha^2 p_4\lambda p_5\lambda p_3\lambda[2-\\
&2q][p_2\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\,\beta\gamma^2 + 42\alpha^2 p_4\lambda[2-2q][p_3\lambda+q\mu_2][p_2\lambda+\\
&q\mu_2]p_5\lambda\,(\alpha+\mu_1){\theta_1}^2{\theta_2}^2\,\beta\gamma^2 + 42\alpha^2 p_4\lambda\,[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]p_5\lambda[2-2q](\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\,\beta\gamma^2\end{aligned}$$

$$\begin{aligned}A_3 = {}&96\alpha\mu_1 p_1\lambda p_4\lambda p_5\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]\theta_1{\theta_2}^2\beta^2\gamma^2 + 42\alpha\mu_1[p_3\lambda+\\
&q\mu_2]p_5\lambda p_4\lambda[2-\theta_1][p_2\lambda+q\mu_2]^2(\alpha+\mu_1){\theta_2}^2\beta^2\gamma^2 + 96\alpha\mu_1 p_4\lambda[p_2\lambda+\\
&q\mu_2]p_5\lambda p_1\lambda[p_3\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\beta^2\gamma + 96\alpha^2[p_5\lambda]^2[p_2\lambda+q\mu_2]p_4\lambda[p_3\lambda+\\
&q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\beta^2\gamma + 42\alpha^2[2-\gamma]p_4\lambda[p_3\lambda+q\mu_2]^2 p_5\lambda[p_2\lambda+q\mu_2](\alpha+\\
&\mu_1){\theta_1}^2{\theta_2}^2\beta^2 + 96\alpha r\mu_1 p_1\lambda p_4\lambda p_5\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]{\theta_1}^2\theta_2\beta^2\gamma^2 +\\
&96\alpha r\mu_1 p_4\lambda p_5\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]^2(\alpha+\mu_1)\theta_1\,\theta_2\beta^2\gamma^2 + 42\alpha r\mu_1[2-\\
&\theta_2]p_5\lambda[p_4\lambda]^2[p_2\lambda+q\mu_2][p_3\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2\beta^2\gamma^2 +\\
&96\alpha\mu_1 p_1\lambda p_4\lambda p_5\lambda[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2]{\theta_1}^2{\theta_2}^2\,\beta\gamma^2 + 42\alpha^2[2-\\
&\beta]p_4\lambda[p_5\lambda]^2[p_3\lambda+q\mu_2][p_2\lambda+q\mu_2](\alpha+\mu_1){\theta_1}^2{\theta_2}^2\,\gamma^2\end{aligned}$$

5. Variance number of customers in the queue for vacation policy I

$$V = H''(1) + H'(1) - [H'(1)^2] \tag{3.3}$$

$$H''(z) = H''(1) = \frac{91N^V}{1260}$$

6. Variance number of customers in the queue for vacation policy II

$$V = H''(1) + H'(1) - [H'(1)^2] \tag{3.3a}$$

$$H''(z) = H''(1) = \frac{91N^V}{1260}$$

The calculated values are tabulated in the table 4.1 and 4.2

Table 4.1: Idle Probabilities

($\alpha = 4.99, \beta = 3.16, \gamma = 0.5, \theta_1 = 0.8, \theta_2 = 0.7, q = 0.07, p = 0.09, \mu_1 = 3.09, \mu_2 = 3.08, r = 0.09, p_0 = 0.99, p_1 = 0.89, p_2 = 0.79, p_3 = 0.69, p_4 = 0.59, p_5 = 0.49$)

λ	Q	
	Model I	Model II
1.5	0.03379	0.02626
1.6	0.03461	0.02681
1.7	0.03557	0.02746
1.8	0.03667	0.02820
1.9	0.03791	0.02904
2.0	0.03932	0.02999
2.1	0.04090	0.03105
2.2	0.04269	0.03223
2.3	0.04471	0.03356
2.4	0.04699	0.03504

In table 4.1, for various arrival rates, the idle probabilities are calculated. As the arrival rate increases, the frequency of server becomes idle also increases.

Table 4.2: Performance measures

($\alpha = 4.99, \beta = 3.16, \gamma = 0.5, \theta_1 = 0.8, \theta_2 = 0.7, q = 0.07, p = 0.09, \mu_1 = 3.09, \mu_2 = 3.08, r = 0.09, p_0 = 0.99, p_1 = 0.89, p_2 = 0.79, p_3 = 0.69, p_4 = 0.59, p_5 = 0.49$)

Model I			Model II		
λ	M	V	λ	M	V
1.5	0.96197	404.52151	1.5	0.96987	408.96355
1.6	0.96120	560.56989	1.6	0.96946	564.26074
1.7	0.96031	765.91132	1.7	0.96896	768.33545
1.8	0.95927	1033.61475	1.8	0.96837	1033.06274
1.9	0.95810	1379.88562	1.9	0.96771	1373.61023
2.0	0.95677	1824.84790	2.0	0.96695	1808.58057
2.1	0.95527	2393.56250	2.1	0.96609	2360.81982
2.2	0.95358	3117.35913	2.2	0.96513	3058.45361
2.3	0.95167	4035.62402	2.3	0.96405	3936.22241
2.4	0.94950	5198.19922	2.4	0.96283	5037.24707

6. Control Chart Analysis

Statistical process control is very much useful in studying quality control of a system.

Many methods are proposed and analyzed by researchers; one is using control chart analysis. A few common characteristics of control chart analysis, whatever may be the type, contains upper and lower control limits. The quality of the data is measured using the limit values. In this analysis there is a control line, which is usually considered to be the target value. Statistically controlled system, the observations lie nearer to the control limit (CL) and within the Upper control limit (UCL) and Lower control limit (LCL). The upper control limit and the lower control limit are measured by the following formulas:

$$CL = M$$

$$UCL = M + 3\sqrt{V}$$

$$LCL = M - 3\sqrt{V}$$

For our model the control limits are calculated for mean number of customers in the system by assuming the batch size follows decapitated geometric distribution

$$a_k = \alpha(1-\alpha)^{k-1}, k = 0,1,2 \ldots .0 < \alpha < 1$$

The first moment is $\frac{1}{\alpha}$ and the second moment is $\frac{2(1-\alpha)}{\alpha^2}$

The calculated values are tabulated in the table 5.1

Table 5.1: Control limits

($\alpha = 4.99, \beta = 3.16, \gamma = 0.5, \theta_1 = 0.8, \theta_2 = 0.7, q = 0.07, p = 0.09, \mu_1 = 3.09, \mu_2 = 3.08, r = 0.09, p_0 = 0.99, p_1 = 0.89, p_2 = 0.79, p_3 = 0.69, p_4 = 0.59, p_5 = 0.49$)

Model I				Model II			
λ	CL	UCL	LCL	λ	CL	UCL	LCL
1.5	0.96197	61.30013	0	1.5	0.96987	61.60140	0
1.6	0.96120	71.99026	0	1.6	0.96946	72.23198	0
1.7	0.96031	83.98561	0	1.7	0.96896	84.12555	0
1.8	0.95927	97.40891	0	1.8	0.96837	97.39225	0
1.9	0.95810	112.39854	0	1.9	0.96771	112.15445	0
2.0	0.95677	129.11150	0	2.0	0.96695	128.54918	0
2.1	0.95527	147.72743	0	2.1	0.96609	146.73090	0
2.2	0.95358	168.45354	0	2.2	0.96513	166.87500	0
2.3	0.95167	191.53134	0	2.3	0.96405	189.18202	0
2.4	0.94950	217.24512	0	2.4	0.96283	213.88351	0

The table 5.1, shows the control limit for various values of arrival rate. For lower limit, the calculated values are negative, in terms the values are taken as zero.

7. Conclusion

In this paper a single server batch arrival queue has been considered. In addition the server takes compulsory vacation during service, the server may breakdown. Except inter arrival time and inter breakdown period random variable all other random variables in this model are generally distributed. That is the inter arrival time and inter breakdown period are negative exponential distributions. The model is analyzed in steady state by applying probability generation function method. In addition statistical quality control process is carried out by the way of control chart analysis for mean number of customers in the system. Some numerical illustrated are obtained. The model can be extended by assuming general inter arrival time and/are generally distributed inter breakdown period.

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