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Operations on Picture Fuzzy Rough Matrices for MCDM Applications

V. S. Subha[1] and G. Dhanusha[2]

[1] *PG and Research department of Mathematics, Govt. Arts College, C.Mutlur, Chidambaram.*

[2] *Department of Mathematics, Annamalai University, Annamalai Nagar, India-608002.*

Abstract:- This work presents the picture fuzzy rough matrix, which generalizes the intuitionistic fuzzy rough matrix. We define picture fuzzy rough matrix and explore its structure, including operations such as AND, OR, min-min composition and the complement. We establish several key properties such as commutativity, associativity, distributivity, idempotency and absorption.

Keywords: *Picture fuzzy rough matrix, fuzzy matrix, picture fuzzy matrix.*

1. Introduction

The notion of fuzzy set theory was established by L.A.Zadeh [12] in the year 1965, which allows for the representation of imprecise information by permitting elements to possess partial memberships within a set. In 1982, Zdzislaw Pawlak [8] developed rough set theory as a mathematical tool for the analysis of imprecise, uncertain and incomplete information. D.Dubois and H.Prade [5] formulated the fuzzy rough set in 1990 by merging fuzzy set theory with rough set theory. In 1986, K.Atanassov [2] presented intuitionistic fuzzy set theory, extending the classical fuzzy framework by assigning both membership and non-membership degrees to each element. Picture fuzzy sets were introduced by B.C.Cuong and V.Kreinovich [3] in 2013, extending intuitionistic fuzzy sets by including a neutral degree to model acceptance, rejection and indifference.

In 1977, M.G.Thomson [11], represented these concepts in matrix form, a fuzzy matrix with values of elements lying in [0,1], representing degrees of membership rather than mere binary classifications. In 2002, M.Pal, S.K.Khan, Amiya K Shyamd [7] introduced the intuitionistic fuzzy matrix which builds on the idea of including both membership and non-membership degrees. S.Dogra and M.Pal [4] proposed Picture fuzzy matrices in 2020, defining them through four degrees- acceptance, rejection, neutrality and refusal. In contrast, rough matrices provide a structured approach to address uncertainty and ambiguity in data.

Moreover, advanced structures such as intuitionistic fuzzy rough matrices [1] merge intuitionistic fuzzy sets with rough set theory to simultaneously illustrate both uncertainty and vagueness. Additionally, picture fuzzy rough matrices integrate picture fuzzy sets and rough set theory to effectively manage the intricate uncertainties that include acceptance rejection, neutrality and vagueness in data sets.

2. Preliminaries

Definition 2.1. [11] A matrix P = $[P_{ij}]_{l\times m}$ is referred to as a fuzzy matrix if each entry P_{ij} assumes a value in [0, 1], representing a membership degree.

Definition 2.2. [7] Consider P = $(\alpha_{ij}, \lambda_{ij})$ as intuitionistic fuzzy matrix (IFM) of dimension $l \times m$. For each ij^{th} entry, α_{ij} and λ_{ij}, taking values in the interval [0,1], represent the corresponding membership and non-membership degrees. These components adhere to the constraint.

$0 \leq \alpha_{ij} + \lambda_{ij} \leq 1$, for all i, j.

Definition 2.3. [4] Assume P = $(\alpha_{ij}, \lambda_{ij}, \beta_{ij})$ denote a picture fuzzy matrix of order $l \times m$. For the ij^{th} element of P, α_{ij}, β_{ij} and λ_{ij} represent the degrees of membership, neutrality and non-membership, respectively, with each value belonging to the interval [0, 1]. These values must satisfy the condition $0 \leq \alpha_{ij} + \lambda_{ij} + \beta_{ij} \leq 1$, for all i and j.

Definition 2.4. [1] Assume Ŭ is a non-empty finite domain of discourse. A picture fuzzy relation $\mathfrak{R}$ on Ŭ is described as a picture fuzzy subset of $\breve{U} \times \breve{U}$ is given by

$$\mathfrak{R} = \{(l,m), \alpha_{ij}(l,m), \beta_{ij}(l,m), \lambda_{ij}(l,m) \mid (l,m) \in (\breve{U} \times \breve{U})\}$$

where $\alpha_{ij}: l \times m \rightarrow [0,1], \beta_{ij}: l \times m \rightarrow [0,1], \lambda_{ij}: l \times m \rightarrow [0,1]$. The membership, neutral and non-membership measures of (l, m) lie in the interval [0, 1].

$0 \leq \alpha_{ij}(l,m) + \beta_{ij}(l,m) + \lambda_{ij}(l,m) \leq 1$, for every $(l,m) \in (\breve{U} \times \breve{U})$.

Definition 2.5. Suppose $\mathfrak{R}$ is a picture fuzzy relation defined over $\breve{U} \times \breve{U}$. The framework $(\breve{U}, \mathfrak{R})$ is termed as picture fuzzy rough approximation framework. For every P belonging to the picture fuzzy power set of Ŭ, the corresponding lower and upper approximations are defined as below.

$$\alpha_{\underline{\mathfrak{R}}(P)}(l) = \bigwedge_{m \in \breve{U}} \left[\alpha_{\underline{\mathfrak{R}}(l,m)} \bigwedge \alpha p(m)\right]$$

$$\beta_{\underline{\mathfrak{R}}(P)}(l) = \bigvee_{m \in \breve{U}} \left[\beta_{\underline{\mathfrak{R}}(l,m)} \bigvee \beta p(m)\right]$$

$$\lambda_{\underline{\mathfrak{R}}(P)}(l) = \bigvee_{m \in \breve{U}} \left[\lambda_{\underline{\mathfrak{R}}(l,m)} \bigvee \lambda p(m)\right]$$

$$\alpha_{\overline{\mathfrak{R}}(P)}(l) = \bigvee_{m \in \breve{U}} \left[\alpha_{\underline{\mathfrak{R}}(l,m)} \bigvee \alpha p(m)\right]$$

$$\beta_{\overline{\mathfrak{R}}(P)}(l) = \bigwedge_{m \in \breve{U}} \left[\beta_{\overline{\mathfrak{R}}(l,m)} \bigwedge \beta p(m)\right]$$

$$\lambda_{\overline{\mathfrak{R}}(P)}(l) = \bigwedge_{m \in \breve{U}} \left[\lambda_{\overline{\mathfrak{R}}(l,m)} \bigwedge \lambda p(m)\right]$$

The Picture fuzzy rough set corresponding to P is given by the pair $(\underline{\mathfrak{R}}(P), \overline{\mathfrak{R}}(P))$, which is denoted by $\mathfrak{R}(P)$.

3. Picture Fuzzy Rough Matrix

Definition 3.1. Let $\breve{U} = \{m_1, m_2, \dots, m_p\}$ be a finite universe of discourse. For j = 1, 2, ..., q, let $\mathfrak{B}_{\sim}(P_j)$ represent the picture fuzzy rough sets defined over Ŭ. The corresponding picture fuzzy rough matrix of P_j is a matrix of order l × m, denoted by, $\mathfrak{B}_{\sim} = \left(\underline{\mathfrak{B}_{\sim}}, \overline{\mathfrak{B}_{\sim}}\right)_{l \times m}$ where $\mathfrak{B}_{\sim} = b_{ij}$, for each i = 1, 2, ..., l and j = 1, 2, ..., m, the corresponding entry is defined as,

$$b_{ij} = \left(\alpha_{\underline{b}_{ij}}, \alpha_{\overline{b}_{ij}}\right), \left(\beta_{\underline{b}_{ij}}, \beta_{\overline{b}_{ij}}\right), \left(\lambda_{\underline{b}_{ij}}, \lambda_{\overline{b}_{ij}}\right), \text{ for all } i, j.$$

The lower and upper approximations of degrees of membership are represented by $\left(\alpha_{\underline{b}_{ij}}, \alpha_{\overline{b}_{ij}}\right)$, the lower and upper approximations of degrees of neutrality are represented by $\left(\beta_{\underline{b}_{ij}}, \beta_{\overline{b}_{ij}}\right)$ and the lower and upper approximations of degrees of non-membership are represented by $\left(\lambda_{\underline{b}_{ij}}, \lambda_{\overline{b}_{ij}}\right)$ that satisfy the condition. $0 \leq \alpha_{\underline{b}_{ij}} + \beta_{\underline{b}_{ij}} + \lambda_{\underline{b}_{ij}} \leq 1$ and $0 \leq \alpha_{\overline{b}_{ij}} + \beta_{\overline{b}_{ij}} + \lambda_{\overline{b}_{ij}} \leq 1$.

Definition 3.2. A picture fuzzy rough matrix of size l × m is termed a Picture fuzzy rough null matrix if every entry of the matrix is given by ((0, 0), (1, 1), (1, 1)). The null matrix is denoted by $\mathcal{N}$.

Definition 3.3. A picture fuzzy rough matrix of order l × m is referred to as a Picture fuzzy rough absolute matrix if all its entries are equal to ((1, 1), (0, 0), (0, 0)). This matrix is denoted by $\mathfrak{A}$.

Definition 3.4. A pair of picture fuzzy rough matrices $\mathfrak{B}_{\sim} = b_{ij}$ and $\mathfrak{C}_{\sim} = c_{ij}$ of the same order are equal iff

$$\alpha_{\underline{b}_{ij}} = \alpha_{\underline{c}_{ij}}, \alpha_{\overline{b}_{ij}} = \alpha_{\overline{c}_{ij}},$$

$$\beta_{\underline{b}_{ij}} = \beta_{\underline{c}_{ij}}, \beta_{\overline{b}_{ij}} = \beta_{\overline{c}_{ij}},$$

$$\lambda_{\underline{b}_{ij}} = \lambda_{\underline{c}_{ij}}, \lambda_{\overline{b}_{ij}} = \lambda_{\overline{c}_{ij}}.$$

4. Operations on Picture fuzzy rough matrices

From now on we denote Picture Fuzzy Rough Matrices as PFRM’s.

Definition 4.1. The AND operator between two PFRM’s $\mathfrak{B}_{\sim}$ and $\mathfrak{C}_{\sim}$ is represented by,

$$\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \left[\min\left(\alpha_{\underline{b}_{ij}}, \alpha_{\underline{c}_{ij}}\right), \min\left(\alpha_{\overline{b}_{ij}}, \alpha_{\overline{c}_{ij}}\right)\right],$$

$$= \left[\max\left(\beta_{\underline{b}_{ij}}, \beta_{\underline{c}_{ij}}\right), \max\left(\beta_{\overline{b}_{ij}}, \beta_{\overline{c}_{ij}}\right)\right]$$

$$= \left[\max\left(\lambda_{\underline{b}_{ij}}, \lambda_{\underline{c}_{ij}}\right), \max\left(\lambda_{\overline{b}_{ij}}, \lambda_{\overline{c}_{ij}}\right)\right].$$

Example 4.2. Consider the PFRM’s corresponding to the universal set $\breve{U} = \{\breve{U}_1, \breve{U}_2\}$ and the attribute set $\underset{\cdot}{\breve{A}} = \{\underset{\cdot}{\breve{A}}_1, \underset{\cdot}{\breve{A}}_2\}$ is given by,

$$\mathfrak{B}_{\sim} = \begin{array}{c} \\ \breve{U}_1 \\ \breve{U}_2 \end{array} \begin{bmatrix} \underset{\cdot}{\breve{A}}_1 & \underset{\cdot}{\breve{A}}_2 \\ (0.2, 0.3)(0.4, 0.06)(0.3, 0.5) & (0.6, 0.2)(0.1, 0.7)(0.3, 0.1) \\ (0.4, 0.2)(0.1, 0.4)(0.2, 0.08) & (0.3, 0.2)(0.06, 0.2)(0.5, 0.2) \end{bmatrix}$$

The PFRM corresponding to the universal set $\breve{U} = \{\breve{U}_1, \breve{U}_2\}$ and the attribute set $\breve{B} = \{\breve{B}_1, \breve{B}_2\}$ is given by,

$$\mathfrak{C}_{\sim} = \begin{array}{c} \\ \breve{U}_1 \\ \breve{U}_2 \end{array} \begin{bmatrix} \breve{B}_1 & \breve{B}_2 \\ (0.1, 0.2)(0.3, 0.2)(0.4, 0.1) & (0.2, 0.4)(0.2, 0.3)(0.4, 0.1) \\ (0.5, 0.3)(0.2, 0.6)(0.3, 0.07) & (0.3, 0.1)(0.05, 0.2)(0.6, 0.3) \end{bmatrix}$$

Now,

$$\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \begin{bmatrix} (0.1, 0.2)(0.4, 0.2)(0.4, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.4, 0.1) \\ (0.4, 0.2)(0.2, 0.6)(0.3, 0.08) & (0.3, 0.1)(0.06, 0.2)(0.6, 0.3) \end{bmatrix}.$$

Definition 4.3. The OR operator between two PFRM’s $\mathfrak{B}_{\sim}$ and $\mathfrak{C}_{\sim}$ is represented by,

$$\mathfrak{B}_{\sim} \bigvee \mathfrak{C}_{\sim} = \left[\max\left(\alpha_{\underline{b}_{ij}}, \alpha_{\underline{c}_{ij}}\right), \max\left(\alpha_{\overline{b}_{ij}}, \alpha_{\overline{c}_{ij}}\right)\right]$$

$$= \left[\min\left(\beta_{\underline{b}_{ij}}, \beta_{\underline{c}_{ij}}\right), \min\left(\beta_{\overline{b}_{ij}}, \beta_{\overline{c}_{ij}}\right)\right]$$

$$= \left[\min\left(\lambda_{\underline{b}_{ij}}, \lambda_{\underline{c}_{ij}}\right), \min\left(\lambda_{\overline{b}_{ij}}, \lambda_{\overline{c}_{ij}}\right)\right].$$

Example 4.4. Consider the PFRM’s $\mathfrak{B}_{\sim}$ and $\mathfrak{C}_{\sim}$ as in Example 4.2.

$$\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \begin{bmatrix} (0.2,0.3)(0.3,0.06)(0.3,0.1) & (0.6,0.4)(0.1,0.3)(0.3,0.1) \\ (0.5, 0.3)(0.1, 0.4)(0.2, 0.07) & (0.3,0.2)(0.05,0.2)(0.5,0.2) \end{bmatrix}.$$

Definition 4.5. The max-min composition of the two PFRM’s $\mathfrak{B}_{\sim p\times q}$ and $\mathfrak{C}_{\sim p\times q}$ denoted by $\mathfrak{B}_{\sim} * \mathfrak{C}_{\sim}$, which is defined as,

$$\mathfrak{B}_{\sim} * \mathfrak{C}_{\sim} = \mathfrak{D}_{\sim p\times r}$$

$$= \left[\max_{j}\left(\alpha_{\underline{b}_{ij}} \wedge \alpha_{\underline{c}_{jk}}\right), \max_{j}\left(\alpha_{\overline{b}_{ij}} \wedge \alpha_{\overline{c}_{jk}}\right)\right]$$

$$\left[\min_{j}\left(\beta_{\underline{b}_{ij}} \vee \beta_{\underline{c}_{jk}}\right), \min_{j}\left(\beta_{\overline{b}_{ij}} \vee \beta_{\overline{c}_{jk}}\right)\right]$$

$$\left[\min_{j}\left(\lambda_{\underline{b}_{ij}} \vee \lambda_{\underline{c}_{jk}}\right), \min_{j}\left(\lambda_{\overline{b}_{ij}} \vee \lambda_{\overline{c}_{jk}}\right)\right]$$

Definition 4.6. The Picture fuzzy rough complement of $\mathfrak{B}_{\sim}$ is given by $\mathfrak{B}'_{\sim} = b'_{ij}$, such that $b'_{ij} = \left[\left(\lambda_{\underline{b}_{ij}}, \lambda_{\overline{b}_{ij}}\right), \left(\beta_{\underline{b}_{ij}}, \beta_{\overline{b}_{ij}}\right), \left(\alpha_{\underline{b}_{ij}}, \alpha_{\overline{b}_{ij}}\right)\right]$.

Theorem 4.7. The commutative property holds for PFRM's over the universe Ŭ, which can be expressed as:

i. $\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \wedge \mathfrak{B}_{\sim}$.
ii. $\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \vee \mathfrak{B}_{\sim}$.

Proof:

i. $\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \min\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \max\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \max\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)$
$= \min\left(\alpha_{c_{ij}}, \alpha_{b_{ij}}\right), \max\left(\beta_{c_{ij}}, \beta_{b_{ij}}\right), \max\left(\lambda_{c_{ij}}, \lambda_{b_{ij}}\right)$
$= \mathfrak{C}_{\sim} \wedge \mathfrak{B}_{\sim}$.
ii. $\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \max\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \min\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \min\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)$
$= \max\left(\alpha_{c_{ij}}, \alpha_{b_{ij}}\right), \min\left(\beta_{c_{ij}}, \beta_{b_{ij}}\right), \min\left(\lambda_{c_{ij}}, \lambda_{b_{ij}}\right)$
$= \mathfrak{C}_{\sim} \vee \mathfrak{B}_{\sim}$.

Example 4.8. Consider the PFRM's that satisfy the commutative property as in Example 4.2.

i. $\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \wedge \mathfrak{B}_{\sim}$.
ii. $\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \vee \mathfrak{B}_{\sim}$.

Proof:

i. $\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \begin{bmatrix} (0.1, 0.2)(0.4, 0.2)(0.4, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.4, 0.1) \\ (0.4, 0.2)(0.2, 0.6)(0.3, 0.08) & (0.3, 0.1)(0.06, 0.2)(0.6, 0.3) \end{bmatrix}$
$= \mathfrak{C}_{\sim} \wedge \mathfrak{B}_{\sim}$.

$\therefore \mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \wedge \mathfrak{B}_{\sim}$.

ii. Similarly, for $\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \mathfrak{C}_{\sim} \vee \mathfrak{B}_{\sim}$.

Theorem 4.9. The associative property holds for PFRM's over the universe Ŭ, which can be expressed as:

(i) $(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim})$.
(ii) $(\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \vee \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim})$.

Proof:

(i) $(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \left[\min\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \max\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \max\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right] \wedge \left[\left(\alpha_{d_{ij}}, \beta_{d_{ij}}, \lambda_{d_{ij}}\right)\right]$
$= \left[\min\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \max\left(\beta_{b_{ij}}, \beta_{c_{ij}}, \beta_{d_{ij}}\right), \max\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$
$= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right]$
$\wedge \left[\min\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \max\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right), \max\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$
$= \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim})$.
(ii) $(\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \vee \mathfrak{D}_{\sim} = \left[\max\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \min\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \min\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right] \vee \left[\left(\alpha_{d_{ij}}, \beta_{d_{ij}}, \lambda_{d_{ij}}\right)\right]$

$$= \left[\max\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \min\left(\beta_{b_{ij}}, \beta_{c_{ij}}, \beta_{d_{ij}}\right), \min\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right]$$
$$\vee \left[\max\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \min\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right), \min\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}).$$

Example 4.10. Consider the PFRM's that satisfy the associative property as in example 4.2.

(i) $(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim})$.
(ii) $(\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \vee \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim})$.

(i) Now $(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim}$

$$(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) = \begin{bmatrix} (0.1, 0.2)(0.4, 0.2)(0.4, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.4, 0.1) \\ (0.4, 0.2)(0.2, 0.6)(0.3, 0.08) & (0.3, 0.1)(0.06, 0.2)(0.6, 0.3) \end{bmatrix}.$$

$$(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \begin{bmatrix} (0.1, 0.1)(0.4, 0.6)(0.4, 0.5) & (0.2, 0.2)(0.4, 0.7)(0.4, 0.1) \\ (0.08, 0.2)(0.2, 0.6)(0.3, 0.2) & (0.07, 0.1)(0.09, 0.3)(0.6, 0.3) \end{bmatrix} \to (1).$$

$$\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim} = \begin{bmatrix} (0.1, 0.1)(0.3, 0.6)(0.4, 0.1) & (0.2, 0.3)(0.4, 0.5)(0.4, 0.1) \\ (0.08, 0.2)(0.2, 0.6)(0.3, 0.2) & (0.07, 0.1)(0.09, 0.3)(0.6, 0.3) \end{bmatrix}.$$

$$\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = \begin{bmatrix} (0.1, 0.1)(0.4, 0.6)(0.4, 0.5) & (0.2, 0.2)(0.4, 0.7)(0.4, 0.1) \\ (0.08, 0.2)(0.2, 0.6)(0.3, 0.2) & (0.07, 0.1)(0.09, 0.3)(0.6, 0.3) \end{bmatrix} \to (2).$$

From equation (1) and (2), we get $(\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim})$.

$\therefore (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \wedge \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim})$.

(ii) Similarly, for $(\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \vee \mathfrak{D}_{\sim} = \mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim})$.

Theorem 4.11. The distributive property holds for PFRM's over the universe Ŭ, which can be expressed as:

(i) $\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim})$.
(ii) $\mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{D}_{\sim})$.

Proof:

(i)
$$\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right] \wedge \left[\max\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \min\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right), \min\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \left[\left(\alpha_{b_{ij}}\right) \wedge \max\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right)\right], \left[\left(\beta_{b_{ij}}\right) \vee \min\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right)\right],$$
$$\left[\left(\lambda_{b_{ij}}\right) \vee \min\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \max\left[\left(\alpha_{b_{ij}} \wedge \alpha_{c_{ij}}\right), \left(\alpha_{b_{ij}} \wedge \alpha_{d_{ij}}\right)\right], \min\left[\left(\beta_{b_{ij}} \vee \beta_{c_{ij}}\right), \left(\beta_{b_{ij}} \vee \beta_{d_{ij}}\right)\right],$$
$$\min\left[\left(\lambda_{b_{ij}} \vee \lambda_{c_{ij}}\right), \left(\lambda_{b_{ij}} \vee \lambda_{d_{ij}}\right)\right]$$
$$= (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}).$$

$\therefore \mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim})$.

(ii)
$$\mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right] \vee \left[\min\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right), \max\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right), \max\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \left[\left(\alpha_{b_{ij}}\right) \vee \min\left(\alpha_{c_{ij}}, \alpha_{d_{ij}}\right)\right], \left[\left(\beta_{b_{ij}}\right) \wedge \max\left(\beta_{c_{ij}}, \beta_{d_{ij}}\right)\right],$$
$$\left[\left(\lambda_{b_{ij}}\right) \wedge \max\left(\lambda_{c_{ij}}, \lambda_{d_{ij}}\right)\right]$$
$$= \min\left[\left(\alpha_{b_{ij}} \vee \alpha_{c_{ij}}\right), \left(\alpha_{b_{ij}} \vee \alpha_{d_{ij}}\right)\right], \max\left[\left(\beta_{b_{ij}} \wedge \beta_{c_{ij}}\right), \left(\beta_{b_{ij}} \wedge \beta_{d_{ij}}\right)\right],$$

$$\max\left[\left(\lambda_{b_{ij}} \wedge \lambda_{c_{ij}}\right), \left(\lambda_{b_{ij}} \wedge \lambda_{d_{ij}}\right)\right]$$
$$= (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{D}_{\sim}).$$

$\therefore \mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{D}_{\sim}).$

Example 4.12. Consider the PFRM's that satisfy the distributive property as in Example 4.2.

(i) $\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}).$
(ii) $\mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{D}_{\sim}).$

(i) Now $\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim})$

$$\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim} = \begin{bmatrix} (0.5, 0.2)(0.3, 0.2)(0.07, 0.1) & (0.2, 0.4)(0.2, 0.3)(0.07, 0.1) \\ (0.5, 0.3)(0.1, 0.4)(0.2, 0.07) & (0.3, 0.5)(0.09, 0.3)(0.6, 0.3) \end{bmatrix}.$$

$$\mathfrak{B}_{\sim} \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = \begin{bmatrix} (0.2, 0.2)(0.4, 0.2)(0.3, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.3, 0.1) \\ (0.4, 0.2)(0.1, 0.4)(0.2, 0.08) & (0.3, 0.2)(0.09, 0.3)(0.6, 0.3) \end{bmatrix} \rightarrow (3).$$

Now, $(\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim})$

$$\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim} = \begin{bmatrix} (0.1, 0.2)(0.4, 0.2)(0.4, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.4, 0.1) \\ (0.4, 0.2)(0.2, 0.6)(0.3, 0.08) & (0.3, 0.1)(0.06, 0.2)(0.6, 0.3) \end{bmatrix}.$$

$$\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim} = \begin{bmatrix} (0.2, 0.1)(0.4, 0.6)(0.3, 0.5) & (0.2, 0.2)(0.4, 0.7)(0.3, 0.1) \\ (0.08, 0.2)(0.1, 0.4)(0.2, 0.2) & (0.07, 0.2)(0.09, 0.3)(0.6, 0.2) \end{bmatrix}.$$

$$(\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) = \begin{bmatrix} (0.2, 0.2)(0.4, 0.2)(0.3, 0.5) & (0.2, 0.2)(0.2, 0.7)(0.3, 0.1) \\ (0.4, 0.2)(0.1, 0.4)(0.2, 0.08) & (0.3, 0.2)(0.09, 0.3)(0.6, 0.3) \end{bmatrix} \rightarrow (4).$$

From equation (3) and (4), we get $\wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}).$

$\therefore \wedge (\mathfrak{C}_{\sim} \vee \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}) \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{D}_{\sim}).$

(ii) Similarly, for $\mathfrak{B}_{\sim} \vee (\mathfrak{C}_{\sim} \wedge \mathfrak{D}_{\sim}) = (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{D}_{\sim}).$

Theorem 4.13. The idempotent property holds for PFRM's over the universe Ŭ, which can be expressed as:

(i) $\mathfrak{B}_{\sim} \wedge \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}.$
(ii) $\mathfrak{B}_{\sim} \vee \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}.$

Proof:

(i) $\mathfrak{B}_{\sim} \wedge \mathfrak{B}_{\sim} = \min\left(\alpha_{b_{ij}}, \alpha_{b_{ij}}\right), \max\left(\beta_{b_{ij}}, \beta_{b_{ij}}\right), \max\left(\lambda_{b_{ij}}, \lambda_{b_{ij}}\right)$

$$= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right]$$
$$= \mathfrak{B}_{\sim}$$

(ii) $\mathfrak{B}_{\sim} \vee \mathfrak{B}_{\sim} = \max\left(\alpha_{b_{ij}}, \alpha_{b_{ij}}\right), \min\left(\beta_{b_{ij}}, \beta_{b_{ij}}\right), \min\left(\lambda_{b_{ij}}, \lambda_{b_{ij}}\right)$

$$= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right]$$
$$= \mathfrak{B}_{\sim}$$

Example 4.14. Consider the PFRM's that satisfy the idempotent property as in Example 4.2.

(i) $\mathfrak{B}_{\sim} \wedge \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}.$
(ii) $\mathfrak{B}_{\sim} \vee \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}.$

(i) Now,

$$\mathfrak{B}_{\sim} \wedge \mathfrak{B}_{\sim} = \begin{bmatrix} (0.2,0.3)(0.4,0.06)(0.3,0.5) & (0.6,0.2)(0.1,0.7)(0.3,0.1) \\ (0.4, 0.2)(0.1, 0.4)(0.2, 0.08) & (0.3,0.2)(0.06,0.2)(0.5,0.2) \end{bmatrix}.$$

$\therefore \mathfrak{B}_{\sim} \wedge \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}.$

(ii) Similarly, for $\mathfrak{B}_{\sim} \vee \mathfrak{B}_{\sim} = \mathfrak{B}_{\sim}$.

Theorem 4.15. The absorption property holds for PFRM's over the universe Ŭ, which can be expressed as:

(i) $\mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.
(ii) $\mathfrak{B}_{\sim} \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.

Proof:

(i)

$$\begin{aligned}\mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) &= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right] \wedge \left[\max\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \min\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \min\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right]\\ &= \left[\left(\alpha_{b_{ij}}\right) \wedge \max\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right)\right], \left[\left(\beta_{b_{ij}}\right) \vee \min\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right)\right],\\ &\quad \left[\left(\lambda_{b_{ij}}\right) \vee \min\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right].\\ &= \max\left[\left(\alpha_{b_{ij}} \wedge \alpha_{b_{ij}}\right), \left(\alpha_{b_{ij}} \wedge \alpha_{c_{ij}}\right)\right], \min\left[\left(\beta_{b_{ij}} \vee \beta_{b_{ij}}\right), \left(\beta_{b_{ij}} \vee \beta_{c_{ij}}\right)\right],\\ &\quad \min\left[\left(\lambda_{b_{ij}} \vee \lambda_{b_{ij}}\right), \left(\lambda_{b_{ij}} \vee \lambda_{c_{ij}}\right)\right]\\ &= \mathfrak{B}_{\sim}.\end{aligned}$$

(ii)

$$\begin{aligned}\mathfrak{B}_{\sim} \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) &= \left[\left(\alpha_{b_{ij}}, \beta_{b_{ij}}, \lambda_{b_{ij}}\right)\right] \vee \left[\min\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right), \max\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right), \max\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right]\\ &= \left[\left(\alpha_{b_{ij}}\right) \vee \min\left(\alpha_{b_{ij}}, \alpha_{c_{ij}}\right)\right], \left[\left(\beta_{b_{ij}}\right) \wedge \max\left(\beta_{b_{ij}}, \beta_{c_{ij}}\right)\right],\\ &\quad \left[\left(\lambda_{b_{ij}}\right) \wedge \max\left(\lambda_{b_{ij}}, \lambda_{c_{ij}}\right)\right]\\ &= \min\left[\left(\alpha_{b_{ij}} \vee \alpha_{b_{ij}}\right), \left(\alpha_{b_{ij}} \vee \alpha_{c_{ij}}\right)\right], \max\left[\left(\beta_{b_{ij}} \wedge \beta_{b_{ij}}\right), \left(\beta_{b_{ij}} \wedge \beta_{c_{ij}}\right)\right],\\ &\quad \max\left[\left(\lambda_{b_{ij}} \wedge \lambda_{b_{ij}}\right), \left(\lambda_{b_{ij}} \wedge \lambda_{c_{ij}}\right)\right]\\ &= \mathfrak{B}_{\sim}.\end{aligned}$$

$\therefore \mathfrak{B}_{\sim} \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.

Example 4.16. Consider the PFRM's that satisfy the absorption property as in Example 4.2.

(i) $\mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.
(ii) $\mathfrak{B}_{\sim} \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.

(i) Now, $\mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim})$

$$\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim} = \begin{bmatrix} (0.2, 0.3)(0.3, 0.06)(0.3, 0.1) & (0.6, 0.4)(0.1, 0.3)(0.3, 0.1) \\ (0.5, 0.3)(0.1, 0.4)(0.2, 0.07) & (0.3, 0.2)(0.05, 0.2)(0.5, 0.2) \end{bmatrix}.$$

$$\mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) = \begin{bmatrix} (0.2, 0.3)(0.4, 0.06)(0.3, 0.5) & (0.6, 0.2)(0.1, 0.7)(0.3, 0.1) \\ (0.4, 0.2)(0.1, 0.4)(0.2, 0.08) & (0.3, 0.2)(0.06, 0.2)(0.5, 0.2) \end{bmatrix}.$$

$\therefore \mathfrak{B}_{\sim} \wedge (\mathfrak{B}_{\sim} \vee \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.

(ii) Similarly, for $\mathfrak{B}_{\sim} \vee (\mathfrak{B}_{\sim} \wedge \mathfrak{C}_{\sim}) = \mathfrak{B}_{\sim}$.

5. Application Framework of Picture fuzzy rough matrices

This part explores aspects regarding PFRM in real life problems.

The problem addressed in this study is the evaluation of multiple waste management methods under uncertain and imprecise information to identify the most environmentally beneficial alternative. Since environmental benefits depend on multiple criteria and involve subjective judgments, an appropriate mathematical framework is required. The expected solution is to determine the waste management method that provides the maximum overall environmental benefit by simultaneously considering all relevant criteria.

5.1 Procedure

The algorithm can be executed by following these steps:

(1) Input the PFRM $\mathfrak{G}_{\sim}$ related to the product set $\breve{U} \times \breve{P}$

(2) Input the PFRM $\mathfrak{H}_{\sim}$ related to the product set $\breve{P} \times \breve{Q}$

(3) Compute the resulting matrices by performing the operations $\mathfrak{J}_{\sim} = \mathfrak{G}_{\sim} * \mathfrak{H}_{\sim}$ and

$\mathfrak{J}_{\sim} = \mathfrak{G}_{\sim} * \mathfrak{H}'_{\sim}$.

(4) Compute the values of $\mathfrak{S}_{\sim ik}, \mathfrak{T}_{\sim ik}$ and $\mathfrak{B}_{\sim ik}$ where,

$$\mathfrak{S}_{\sim ik} = \left[\max\left(\alpha_{\bar{i}_{ik}}, \alpha_{\underline{j}_{ik}}\right), \min\left(\beta_{\bar{i}_{ik}}, \beta_{\underline{j}_{ik}}\right), \min\left(\lambda_{\bar{i}_{ik}}, \lambda_{\underline{j}_{ik}}\right)\right]$$

$$\mathfrak{T}_{\sim ik} = \left[\max\left(\alpha_{\underline{i}_{ik}}, \alpha_{\underline{j}_{ik}}\right), \min\left(\beta_{\underline{i}_{ik}}, \beta_{\bar{j}_{ik}}\right), \min\left(\lambda_{\underline{i}_{ik}}, \lambda_{\bar{j}_{ik}}\right)\right]$$

$$\mathfrak{B}_{\sim ik} = \left[\frac{\alpha_{s_{ik}} + \alpha_{t_{ik}}}{2}\right], \left[\frac{\beta_{s_{ik}} - \beta_{t_{ik}}}{2}\right], \left[\frac{\lambda_{s_{ik}} - \lambda_{t_{ik}}}{2}\right]$$

(5) Contrast and result. For every column, the element $\mathfrak{B}_{\sim ik}$ exhibiting the maximum membership degrees, maximum neutrality and the minimum non-membership degree is regarded as the most appropriate representative from the universal set for that column.

5.2 Example

Let us assume that an environmental scientist is analyzing three waste management methods namely $\underset{\cdot}{\breve{A}}_1$ (Recycling), $\underset{\cdot}{\breve{A}}_2$ (Composting) and $\underset{\cdot}{\breve{A}}_3$ (Landfill Management) to identify which one provides the greatest overall environmental benefit. Let the universal set $\breve{U} = \{\check{c}_1, \check{c}_2, \check{c}_3\}$ represent the types of environmental benefits, where

$\check{c}_1$ = Pollution Reduction, $\check{c}_2$ = Resource Conservation, $\check{c}_3$ = Ecosystem Protection.

Step:1 The environmental benefits provided by each waste management method are represented using PFRM $\mathfrak{G}_{\sim}$, an illustrated below:

$$\mathfrak{G}_{\sim} = \begin{array}{c} \\ \underset{\cdot}{\breve{A}}_1 \\ \underset{\cdot}{\breve{A}}_2 \\ \underset{\cdot}{\breve{A}}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \check{C}_1 & \check{C}_2 & \check{C}_3 \end{array} \\ \begin{bmatrix} (0.2, 0.8)(0.03, 0.07)(0.02, 0.05) & (0.4, 0.06)(0.08, 0.6)(0.05, 0.08) & (0.05, 0.01)(0.02,0.6)(0.01,0.2) \\ (0.03, 0.5)(0.08, 0.2)(0.7, 0.1) & (0.3, 0.6)(0.03, 0.1)(0.4, 0.2) & (0.1, 0.4)(0.2, 0.04)(0.4, 0.07) \\ (0.3, 0.3)(0.08, 0.3)(0.4, 0.04) & (0.1, 0.8)(0.01, 0.04)(0.3, 0.4) & (0.2, 0.5)(0.05, 0.06)(0.8, 0.08) \end{bmatrix} \end{array}$$

Let $\breve{U} = \{\check{Z}_1, \check{Z}_2, \check{Z}_3\}$ represent the different environmental zones, each having distinct ecological characteristics, where

$\check{Z}_1$ = Urban Areas, $\check{Z}_2$ = Agricultural Regions, $\check{Z}_3$ = Coastal Ecosystems.

Step:2 The effectiveness of the waste management methods across these environmental zones are represented using a PFRM $\mathfrak{H}_{\sim}$, as shown below:

$$\mathfrak{H}_{\sim} = \begin{array}{c} \\ \check{c}_1 \\ \check{c}_2 \\ \check{c}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \check{Z}_1 & \check{Z}_2 & \check{Z}_3 \end{array} \\ \begin{bmatrix} (0.2, 0.4)(0.04, 0.1)(0.5, 0.02) & (0.8, 0.5)(0.09, 0.03)(0.3, 0.02) & (0.9, 0.4)(0.3, 0.5)(0.6, 0.07) \\ (0.2, 0.3)(0.06, 0.1)(0.2, 0.08) & (0.09, 0.8)(0.3, 0.1)(0.1, 0.1) & (0.3, 0.4)(0.1, 0.6)(0.7, 0.03) \\ (0.1, 0.7)(0.07, 0.3)(0.5, 0.09) & (0.7, 0.5)(0.01, 0.2)(0.1, 0.2) & (0.3, 0.1)(0.03, 0.04)(0.5, 0.08) \end{bmatrix} \end{array}$$

$$\mathfrak{H}'_{\sim} = \begin{array}{c} \\ \check{C}_1 \\ \check{C}_2 \\ \check{C}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \check{Z}_1 & \check{Z}_2 & \check{Z}_3 \end{array} \\ \begin{bmatrix} (0.5, 0.02)(0.04, 0.1)(0.2, 0.4) & (0.3, 0.02)(0.09, 0.03)(0.8, 0.5) & (0.6, 0.07)(0.3, 0.5)(0.9, 0.4) \\ (0.2, 0.08)(0.06, 0.1)(0.2, 0.3) & (0.1, 0.1)(0.3, 0.1)(0.09, 0.8) & (0.7, 0.03)(0.1, 0.6)(0.3, 0.4) \\ (0.5, 0.09)(0.07, 0.3)(0.1, 0.7) & (0.1, 0.2)(0.01, 0.2)(0.7, 0.5) & (0.5, 0.08)(0.03, 0.04)(0.3, 0.1) \end{bmatrix} \end{array}$$

Step:3 The two PFRM's $\mathfrak{G}_{\sim}$ and $\mathfrak{H}_{\sim}$ are derived from the composition of waste management methods and environmental zones. These are represented as $\mathfrak{G}_{\sim} * \mathfrak{H}_{\sim}$ and $\mathfrak{G}_{\sim} * \mathfrak{H}'_{\sim}$ as shown below:

$$\begin{array}{ccc} \check{Z}_1 & \check{Z}_2 & \check{Z}_3 \end{array}$$

$$\mathfrak{I}_{\sim} = \begin{array}{c}\underset{\cdot}{\breve{A}}_1\\ \underset{\cdot}{\breve{A}}_2\\ \underset{\cdot}{\breve{A}}_3\end{array}\begin{bmatrix} (0.2,0.4)(0.04,0.1)(0.2,0.05) & (0.2,0.4)(0.06,0.1)(0.4,0.09) & (0.2,0.5)(0.06,0.1)(0.3,0.04) \\ (0.2,0.5)(0.02,0.07)(0.1,0.05) & (0.1,0.6)(0.09,0.1)(0.4,0.1) & (0.3,0.8)(0.05,0.1)(0.3,0.04) \\ (0.3,0.4)(0.03,0.5)(0.5,0.07) & (0.3,0.4)(0.1,0.04)(0.5,0.08) & (0.3,0.4)(0.05,0.06)(0.6,0.07) \end{bmatrix}$$

$$\mathfrak{I}_{\sim} = \begin{array}{cc} & \begin{array}{ccc}\check{Z}_1 & \check{Z}_2 & \check{Z}_3\end{array} \\ \begin{array}{c}\underset{\cdot}{\breve{A}}_1\\ \underset{\cdot}{\breve{A}}_2\\ \underset{\cdot}{\breve{A}}_3\end{array} & \begin{bmatrix} (0.2,0.06)(0.04,0.1)(0.1,0.3) & (0.2,0.09)(0.06,0.1)(0.4,0.3) & (0.3,0.09)(0.06,0.1)(0.3,0.4) \\ (0.2,0.06)(0.02,0.07)(0.09,0.5) & (0.1,0.2)(0.09,0.1)(0.4,0.5) & (0.3,0.2)(0.05,0.1)(0.3,0.5) \\ (0.4,0.07)(0.03,0.5)(0.3,0.2) & (0.3,0.08)(0.1,0.04)(0.4,0.1) & (0.3,0.08)(0.05,0.06)(0.3,0.1) \end{bmatrix}\end{array}$$

Step:4 Calculation of $\mathfrak{S}_{\sim ik}, \mathfrak{T}_{\sim ik}, \mathfrak{B}_{\sim ik}$

$$\mathfrak{S}_{\sim ik} = \begin{array}{cc} & \begin{array}{ccc}\check{Z}_1 & \check{Z}_2 & \check{Z}_3\end{array} \\ \begin{array}{c}\underset{\cdot}{\breve{A}}_1\\ \underset{\cdot}{\breve{A}}_2\\ \underset{\cdot}{\breve{A}}_3\end{array} & \begin{bmatrix} (0.4,0.04,0.05) & (0.4,0.06,0.09) & (0.5,0.06,0.04) \\ (0.5,0.02,0.05) & (0.6,0.09,0.1) & (0.8,0.05,0.04) \\ (0.4,0.3,0.07) & (0.4,0.04,0.08) & (0.4,0.05,0.07) \end{bmatrix}\end{array}$$

$$\mathfrak{T}_{\sim ik} = \begin{array}{cc} & \begin{array}{ccc}\check{Z}_1 & \check{Z}_2 & \check{Z}_3\end{array} \\ \begin{array}{c}\underset{\cdot}{\breve{A}}_1\\ \underset{\cdot}{\breve{A}}_2\\ \underset{\cdot}{\breve{A}}_3\end{array} & \begin{bmatrix} (0.2,0.04,0.2) & (0.2,0.06,0.3) & (0.3,0.06,0.3) \\ (0.2,0.02,0.1) & (0.1,0.09,0.4) & (0.3,0.05,0.3) \\ (0.4,0.03,0.2) & (0.3,0.04,0.1) & (0.3,0.05,0.1) \end{bmatrix}\end{array}$$

$$\mathfrak{B}_{\sim ik} = \begin{array}{cc} & \begin{array}{ccc}\check{Z}_1 & \check{Z}_2 & \check{Z}_3\end{array} \\ \begin{array}{c}\underset{\cdot}{\breve{A}}_1\\ \underset{\cdot}{\breve{A}}_2\\ \underset{\cdot}{\breve{A}}_3\end{array} & \begin{bmatrix} (0.5,0.02,-0.05) & (0.5,0.03,-0.06) & (0.65,0.03,-0.28) \\ (0.6,0.01,0) & (0.65,0.045,-0.1) & (0.95,0.025,-0.11) \\ (0.6,0.015,-0.03) & (0.55,0.02,0.03) & (0.55,0.02,0.03) \end{bmatrix}\end{array}$$

Step:5 By examining the values in the first column, we can conclude that Composting ($\underset{\cdot}{\breve{A}}_2$) most effective in resource conservation within urban areas. Similarly, Composting ($\underset{\cdot}{\breve{A}}_2$) is also highly beneficial for resource conservation in agricultural regions, while Composting ($\underset{\cdot}{\breve{A}}_2$) also contributes significantly to resource conservation in coastal ecosystems.

6. Conclusion

This work proposes PFRM’s and illustrates their usefulness through practical applications. We have established the definitions and discussed various operators including AND, OR, max-min composition and the Picture fuzzy rough complement. Additionally, fundamental mathematical properties like commutativity, associativity, distributivity, idempotency and absorption were analyzed through appropriate examples. The applicability of PFRM’s is validated through an MCDM case analysis involving an agricultural context.

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