

Free Vibration of Tapered, Damaged, and Functionally Graded Beams with Elastic Supports: A Chebyshev Collocation Study

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Abstract

This paper presents a comprehensive investigation of the free vibration characteristics of beams with variable cross-sections and elastically restrained boundaries using a high-precision Chebyshev collocation method. The methodology employs Chebyshev–Gauss–Lobatto collocation points with cosine transformation and direct analytical differentiation formulas to reduce the governing differential equations to algebraic eigenvalue problems. Three distinct structural configurations are analyzed: (i) linearly tapered beams with flexible ends representing non-ideal structural connections, (ii) beams with exponentially varying properties and damaged boundaries for structural health monitoring applications, and (iii) functionally graded beams with general elastic constraints relevant to advanced material systems. The implementation utilizes multiprecision arithmetic to ensure numerical stability and reliable eigenvalue separation. Extensive validation against three independent analytical and numerical solutions demonstrates exceptional agreement. The method requires only 20–25 collocation points compared to 40–60 nodes typically needed in finite element or differential quadrature methods, while maintaining spectral accuracy. Natural frequencies accurate to six decimal places are presented for various boundary conditions, different taper, varying damage parameters, and different material gradation indices. The results provide valuable benchmark data for structural design optimization, damage detection algorithms, and validation of commercial finite element software.

Keywords: Chebyshev collocation method; variable cross-section beams; elastic boundary conditions; functionally graded materials; damage detection; free vibration analysis

1. Introduction

1.1 Motivation and Background

The analysis of structural vibrations with spatially varying material and geometric properties represents a fundamental problem in engineering mechanics with far-reaching implications across aerospace, civil, mechanical, and marine engineering disciplines. Modern engineering structures increasingly employ variable cross-section beams to achieve optimal structural performance through strategic mass distribution, enhanced stiffness-to-weight ratios, and tailored dynamic characteristics [1–2]. Simultaneously, the advent of advanced manufacturing techniques including additive manufacturing, functionally graded materials (FGMs), and smart material systems has enabled unprecedented control over spatial property variation [3–5]. Furthermore, real engineering connections rarely conform to classical idealized boundary conditions; instead, they exhibit varying degrees of rotational and translational flexibility that significantly influence dynamic behavior [6–7]. The confluence of these factors — variable geometry, material gradation, and non-classical boundaries — necessitates robust computational methods capable of accurately predicting vibrational characteristics while maintaining computational efficiency.

The problem formulation addressed in this study has immediate relevance across multiple engineering disciplines. In aerospace structures, tapered wings, turbine blades, and helicopter rotors require non-uniform cross-sections to achieve optimal strength-to-weight performance and enhanced dynamic stability [8–9]. In civil infrastructure, bridge girders, transmission towers, and building columns are frequently designed with spatially varying stiffness to improve seismic resistance and serviceability performance [10–11]. Within mechanical systems, components such as robot manipulators, precision machine tool elements, and automotive driveshafts incorporate variable geometry to optimize vibration characteristics and dynamic response [12–13]. The formulation is also directly applicable to structural health monitoring, where frequency-based damage detection techniques exploit the sensitivity of modal parameters to stiffness degradation and localized defects [14–15]. In advanced material systems, including functionally graded material (FGM) pressure vessels, thermal barrier coatings, and biomedical implants, spatial tailoring of material properties enables controlled mechanical behavior under complex loading environments [3,16,17]. Furthermore, in foundation engineering, applications such as beams on elastic foundations, pile–soil interaction models, and semi-rigid steel connections necessitate accurate modeling of stiffness variation and boundary interaction effects [18–19].

1.2 Literature Review

1.2.1 Tapered Beams with Classical Boundaries

The vibration analysis of tapered beams has attracted sustained research interest over seven decades. The foundational work of Mabie and Rogers [20–21] systematically analyzed various taper configurations with concentrated masses. The exact analytical approach based on Bessel function theory, pioneered by De Rosa and Auciello [22], provided comprehensive solutions for beams with linearly varying width and depth, deriving frequency equations as functions of taper ratio and boundary conditions. However, the Bessel function methodology becomes increasingly complex for non-classical boundaries and requires careful numerical evaluation of special functions with potential accuracy issues near zeros and singularities.

Recent advances in analytical methods include the Adomian decomposition method applied by Hsu et al. [23] for non-uniform beams with general elastic constraints, the variational iteration method employed by Liu and Gurram [24] demonstrating rapid convergence, and the differential transformation method utilized by Rajasekaran [25] for centrifugally stiffened axially FGM tapered Timoshenko beams. While these semi-analytical techniques offer insights into solution structure, they often require substantial symbolic manipulation and may encounter convergence difficulties for certain parameter ranges.

1.2.2 Numerical Methods for Variable Cross-Section Beams

Numerical approaches have proven particularly effective for complex configurations. The finite element method (FEM) remains the industry standard, with refined formulations by Alshorbagy et al. [26] employing graded finite elements for FGM beams and Chakraborty et al. [27] developing beam elements with polynomial variation of properties. However, FEM typically requires fine mesh discretization (40–100 elements) for high-frequency modes, leading to large system matrices and computational overhead.

The differential quadrature method (DQM), introduced for structural mechanics by Bert and Malik [28] and extensively developed by Shu [29], approximates derivatives using weighted summations at discrete points. Applications to tapered beams include work by Karami and Malekzadeh [30] for thermal buckling, Pradhan and Murmu [31] for beams on elastic foundations, and Huang et al. [32] for rotating tapered Timoshenko beams. While DQM achieves high accuracy with fewer points than FEM (typically 30–40), it requires careful selection of grid distribution and weighting coefficients.

Spectral methods based on orthogonal polynomial expansions offer exponential convergence rates. The Chebyshev polynomial approach was pioneered by Fox and Parker [33] and comprehensively developed by Boyd [34]. Recent implementations include Sari and Butcher [35] using shifted Chebyshev polynomials for damaged boundaries, Zhou et al. [36] applying Chebyshev collocation to thick rectangular plates, and Mao and Pietrzko [37] analyzing stepped beams with Adomian–Chebyshev coupling. The differential transformation method (DTM), based on Taylor series expansion, was introduced by Zhou [38] and applied to beam vibrations

by Malik and Dang [39], with notable contributions by Kaya and Ozgumus [40–41] for coupled flexural–torsional vibrations and Wattanasakulpong and Ungbhakorn [42] for FGM beams with elastic constraints.

1.2.3 Functionally Graded Materials in Structural Applications

Functionally graded materials, characterized by continuous spatial variation of composition and properties, have emerged as a transformative material class. Comprehensive reviews by Jha et al. [43], Thai and Kim [44], and Swaminathan et al. [45] document extensive research on FGM structures. Theoretical frameworks for FGM beam analysis include the elasticity solution by Sankar [46], analytical solutions by Zhong and Yu [47] using Airy stress functions, and higher-order theories by Simsek [48] comparing various beam formulations. Active research directions encompass temperature-dependent properties [49–50], porosity effects [51–52], nonlocal size-dependent formulations [53–54], stochastic and interval uncertainty analysis [55–56], and material distribution optimization [57–58].

1.2.4 Elastic Boundary Conditions and Damage Modeling

Real structural connections exhibit partial restraint between classical idealizations. Pioneering work by Laura et al. [59] established Rayleigh–Ritz solutions for beams with elastic supports, while Wang and Wang [60] derived exact solutions for various spring configurations. Subsequent studies investigated beams on Winkler–Pasternak foundations [61–62], asymmetric and frequency-dependent rotational spring boundaries [63–64], and crack and delamination damage modeled through reduced spring stiffness [65–66]. For FGM structures specifically, elastic boundary effects have been studied by Burlon et al. [67], Yan et al. [68], and Fan and Qiao [69], and remain an active area due to the coupling between material gradation and boundary compliance.

1.2.5 Research Gaps

Despite significant advances, several important gaps remain in the literature. A unified computational framework applicable simultaneously to linearly tapered, exponentially varying, and functionally graded beams with arbitrary elastic boundaries is lacking. Benchmark-quality data with sub-0.01% accuracy are unavailable for many practically relevant configurations. The correct analytical formulas for Chebyshev derivative evaluation at boundary nodes — critical for accuracy — have not been consistently documented. The simultaneous effects of material gradation, geometric taper, and boundary damage on natural frequencies have not been systematically quantified. Furthermore, practical guidelines for the effective spring stiffness ranges that approximate classical boundary conditions remain to be established.

1.3 Research Objectives and Contributions

This study addresses the foregoing gaps by developing a unified Chebyshev collocation framework capable of handling linearly tapered, exponentially varying, and functionally graded beams subject to arbitrary elastic boundary conditions within a single computational formulation. The proposed approach delivers benchmark-quality results with six-decimal precision, thereby providing reliable reference data for future investigations. A comprehensive parametric analysis is performed to systematically examine the influence of taper ratios ranging from 1.0 to 2.0, damage parameters varying from 0 to 100%, power-law indices between 0.2 and 10.0 for functionally graded materials, and a wide spectrum of translational and rotational spring stiffnesses representing elastic supports. The numerical results are rigorously validated against three independent classes of solutions: analytical formulations based on Bessel functions, semi-analytical Chebyshev polynomial methods, and numerical differential transformation techniques. Finally, computational efficiency is demonstrated through detailed convergence studies, highlighting the reduced discretization requirements of the present method in comparison with alternative spectral and transformation-based approaches.

On the numerical side, this work delivers the first comprehensive dataset with six-decimal accuracy for tapered beams with ten distinct elastic boundary combinations, a systematic damage parameter study for exponentially varying beams, and complete frequency spectra for FGM beams with power-law indices spanning two orders of magnitude. On the physical side, it quantifies taper ratio exponents governing frequency scaling, identifies critical damage thresholds, and characterizes the nonlinear FGM response with its transition at unit power-law index.

1.4 Organization of the paper

The paper is organized as follows. Section 2 presents the mathematical formulation including governing equations for the three beam types and general elastic boundary conditions. Section 3 details the Chebyshev collocation methodology with analytical differentiation formulas and eigenvalue problem formulation. Section 4 provides extensive numerical results with comprehensive validation against literature and systematic parametric studies. Section 5 presents the implementation details with Section 6 summarizing the findings.

2. Mathematical Formulation

2.1 General Governing Equation

Consider the free transverse vibration of an Euler–Bernoulli beam with variable cross-section occupying the domain $0 \leq x \leq L$. The governing equation is:

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$

where $w(x, t)$ is the transverse displacement, E is Young's modulus, $I(x)$ is the second moment of area, ρ is the mass density, $A(x)$ is the cross-sectional area, x is the spatial coordinate along the beam axis, t is time, and L is the beam length.

The formulation rests on the following assumptions: (i) Euler–Bernoulli (classical) beam theory, in which plane sections remain plane and perpendicular to the neutral axis; (ii) small deformations with infinitesimal strains and rotations; (iii) linear elastic material behavior; and (iv) negligible shear deformation and rotary inertia, valid when $L/h > 10$.

2.2 Harmonic Solution and Nondimensionalization

Assuming harmonic motion $w(x, t) = W(x) e^{i\omega t}$, where ω is the circular frequency (rad/s) and $W(x)$ is the spatial mode shape, substitution into Eq. (1) yields:

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W}{dx^2} \right] - \rho A(x) \omega^2 W = 0 \quad (2)$$

Introducing the non-dimensional coordinate $z = x/L \in [0, 1]$ and the reference quantities $I_0 = I(0)$ and $A_0 = A(0)$, the nondimensional quantities are defined as:

$$\tilde{W}(z) = \frac{W(Lz)}{L}, \quad \tilde{I}(z) = \frac{I(Lz)}{I_0}, \quad \tilde{A}(z) = \frac{A(Lz)}{A_0} \quad (3)$$

The non-dimensional frequency parameter is $\lambda^4 = \frac{\rho A_0 \omega^2 L^4}{EI_0}$ and the governing equation becomes:

$$\frac{d^2}{dz^2} \left[\tilde{I}(z) \frac{d^2 \tilde{W}}{dz^2} \right] - \lambda^4 \tilde{A}(z) \tilde{W} = 0 \quad (4)$$

For a uniform beam ($\tilde{I} = \tilde{A} = 1$), Eq. (4) reduces to $W^{(4)} - \lambda^4 W = 0$, which admits the expected solutions $W = e^{\pm \lambda z}$, $e^{\pm i\lambda z}$. For brevity, tildes are omitted hereafter with the understanding that all quantities are nondimensional.

2.3 Domain Mapping and Coordinate Systems

The Chebyshev collocation method operates on the standard Chebyshev domain $\xi \in [-1, 1]$. The forward and inverse mappings between the physical domain $z \in [0, 1]$ and the Chebyshev domain are $\xi = 2z - 1$, $z = 1/2(1 + \xi)$. Application of the chain rule yields the derivative scaling relations $\frac{d^m}{dz^m} = 2^m \frac{d^m}{d\xi^m}$

which must be applied consistently in all derivative formulas and boundary conditions.

2.4 Problem-Specific Formulations

2.4.1 Linearly Tapered Beams

For beams with a linearly varying rectangular cross-section, the width and height are

$b(z) = b_0[(\alpha - 1)z + 1]$, $h(z) = h_0[(\alpha - 1)z + 1]$ where $\alpha = b_L/b_0 = h_L/h_0$ is the taper ratio. The cross-sectional area and second moment of area are:

$$A(z) = A_0[(\alpha - 1)z + 1]^2, \quad I(z) = I_0[(\alpha - 1)z + 1]^4 \quad (5)$$

2.4.2 Exponentially Varying Beams

For beams with exponential variation controlled by the gradation parameter β , $I(z) = I_0 e^{4\beta z}$, $A(z) = A_0 e^{2\beta z}$. Expanding the governing equation by applying the product rule to the left hand side of Eq.(4)

yields the explicit form:

$$EI_0 e^{4\beta z} \frac{d^4 W}{dz^4} + 8\beta EI_0 e^{4\beta z} \frac{d^3 W}{dz^3} + 16\beta^2 EI_0 e^{4\beta z} \frac{d^2 W}{dz^2} - \rho A_0 e^{2\beta z} \omega^2 W = 0 \quad (6)$$

2.4.3 Functionally Graded Beams (Power-Law Distribution)

For FGM beams the material composition varies through the beam thickness, not along the axial coordinate $z = x/L$. A dedicated through-thickness coordinate $\zeta \in [-h/2, h/2]$ is introduced, where h is the beam height, kept strictly distinct from the axial coordinate z . The ceramic–metal composition follows a power-law distribution:

$$E(\zeta) = (E_c - E_m) \left(\frac{\zeta}{h} + \frac{1}{2}\right)^n + E_m, \quad \rho(\zeta) = (\rho_c - \rho_m) \left(\frac{\zeta}{h} + \frac{1}{2}\right)^n + \rho_m \quad (7)$$

where subscripts c and m denote ceramic and metal respectively, and $n \geq 0$ is the power-law index. Because E varies through the cross-section, the structural quantities must be replaced by the cross-sectionally homogenized effective bending rigidity and effective mass per unit length:

$$EI_{\text{eff}} = \int_A E(\zeta) \zeta^2 dA = b \int_{-h/2}^{h/2} E(\zeta) \zeta^2 d\zeta \quad (8a)$$

$$(\rho A)_{\text{eff}} = \int_A \rho(\zeta) dA = b \int_{-h/2}^{h/2} \rho(\zeta) d\zeta \quad (8b)$$

For a rectangular cross-section of constant width b , these integrals can be evaluated in closed form for arbitrary n . Since the cross-section is geometrically uniform along the axis, both EI_{eff} and $(\rho A)_{\text{eff}}$ are constants. The nondimensional frequency parameter is accordingly

$\lambda^4 = \frac{(\rho A)_{\text{eff}} \omega^2 L^4}{EI_{\text{eff}}}$ and the governing equation retains the form of Eq. (3) with $\tilde{I}(z) = \tilde{A}(z) = 1$, the dependence on n being fully captured in λ^4 through the ratio $(\rho A)_{\text{eff}}/EI_{\text{eff}}$.

2.5 Boundary Conditions

2.5.1 Classical Boundary Conditions

The moment–curvature and shear force–moment relations are:

$$M(z) = -EI(z) \frac{d^2 W}{dz^2}, \quad Q(z) = -\frac{d}{dz} \left[EI(z) \frac{d^2 W}{dz^2} \right] \quad (9)$$

Table 1. Classical boundary conditions for Euler–Bernoulli beams. For a free end the shear condition is written in its general form valid for variable $EI(z)$; the simplified form $W''' = 0$ holds only when EI is constant.

Boundary Type	Symbol	Displacement	Slope	Moment	Shear
Clamped	C	$W = 0$	$W' = 0$	—	—
Simply Supported	S	$W = 0$	—	$W'' = 0$	—
Free	F	—	—	$W'' = 0$	$(EIW''')' = 0$

2.5.2 Elastically Restrained Boundaries

For beams with translational spring stiffness k_T and rotational spring stiffness k_R at each end, the nondimensional spring parameters are:

$$K_{TL} = \frac{k_{TL}L^3}{EI_0}, \quad K_{RL} = \frac{k_{RL}L}{EI_0}, \quad K_{TR} = \frac{k_{TR}L^3}{EI_0}, \quad K_{RR} = \frac{k_{RR}L}{EI_0} \quad (10)$$

At the left end ($z = 0$), the shear–displacement and moment–slope equilibrium conditions are:

$$-\frac{d}{dz} \left[I(z) \frac{d^2W}{dz^2} \right] \Big|_{z=0} = -K_{TL} W(0) \quad (11)$$

$$-I(0) W''(0) = K_{RL} W'(0) \quad (12)$$

Expanding Eq. (11) using the product rule and dividing through by $I(0)$ gives:

$$W'''(0) = -\frac{I'(0)}{I(0)} W''(0) + \frac{K_{TL}}{I(0)} W(0) \quad (13)$$

At the right end ($z = 1$):

$$W'''(1) = -\frac{I'(1)}{I(1)} W''(1) + \frac{K_{TR}}{I(1)} W(1) \quad (14a)$$

$$-I(1) W''(1) = K_{RR} W'(1) \quad (14b)$$

The spring stiffness limits recover classical conditions: $K_T = 0$ corresponds to a free translation, $K_T \rightarrow \infty$ prevents translation, $K_R = 0$ allows free rotation, and $K_R \rightarrow \infty$ prevents rotation.

2.5.3 Damage Modeling

Boundary damage is modeled through a reduced spring stiffness $k_{\text{damaged}} = k_{\text{intact}}(1 - D)$

where $D \in [0,1]$ is the damage index: $D = 0$ represents an intact boundary, $0 < D < 1$ represents partial damage, and $D = 1$ represents complete failure (zero stiffness, equivalent to a free boundary).

3. Chebyshev Collocation Methodology

3.1 Chebyshev–Gauss–Lobatto Collocation Points

The computational domain $\xi \in [-1,1]$ is discretized using $N + 1$ Chebyshev–Gauss–Lobatto (CGL) points obtained via cosine transformation:

$$\xi_j = -\cos\left(\frac{j\pi}{N}\right), \quad j = 0,1,2, \dots, N \quad (15)$$

These points cluster near the boundaries $\xi = \pm 1$, include the endpoints $\xi_0 = -1$ and $\xi_N = +1$ as collocation points, are optimal for polynomial interpolation (minimizing the Runge phenomenon), and yield a Lebesgue constant that grows only logarithmically with N .

3.2 Analytical Differentiation at Boundary Nodes

A distinctive feature of the present implementation is the use of exact analytical formulas for derivatives of Chebyshev polynomial expansions at the boundary nodes. Given the Chebyshev expansion:

$W(\xi) = \sum_{k=0}^N a_k T_k(\xi)$, where $T_k(\xi) = \cos(k \arccos \xi)$ are Chebyshev polynomials of the first kind, the derivatives at the right boundary ($\xi = +1$) are:

$$T_k'(1) = k^2, \quad T_k''(1) = \frac{k^2(k^2-1)}{3}, \quad T_k'''(1) = \frac{k^2(k^2-1)(k^2-4)}{15}, \quad T_k^{(4)}(1) = \frac{k^2(k^2-1)(k^2-4)(k^2-9)}{105} \quad (16)$$

At the left boundary ($\xi = -1$), parity relations apply:

$$T_k^{(m)}(-1) = (-1)^k T_k^{(m)}(1) \quad (m \text{ even}) \quad (17)$$

$$T_k^{(m)}(-1) = (-1)^{k+1} T_k^{(m)}(1) \quad (m \text{ odd}) \quad (18)$$

3.3 Matrix Assembly and Eigenvalue Problem

The governing differential equation is discretized at interior collocation points $j = 1, \dots, N - 1$, while the boundary conditions are enforced at $j = 0$ and $j = N$. This leads to the generalized eigenvalue problem:

$$[\mathbf{K}]\{\mathbf{W}\} = \lambda^4 [\mathbf{M}]\{\mathbf{W}\} \tag{19}$$

where $[\mathbf{K}]$ is the stiffness matrix assembled from the bending term, $[\mathbf{M}]$ is the mass matrix assembled from the inertia term, $\{\mathbf{W}\}$ is the displacement vector at collocation points, and λ^4 is the eigenvalue corresponding to the nondimensional frequency parameter. The eigenvalue problem (19) is solved using multiprecision arithmetic to ensure reliable separation of closely spaced eigenvalues.

4. Numerical Results and Validation

4.1 Convergence Study

Table 2 presents the convergence of the fundamental nondimensional frequency for a linearly tapered beam with $\alpha = 1.4$ under simply supported boundary conditions as the number of collocation points N is increased. Convergence to six decimal places is achieved at $N = 22$ – 25 points, confirming the spectral accuracy of the method.

Table 2. Convergence of the fundamental frequency Ω_1 with increasing collocation points for a linearly tapered beam ($\alpha = 1.4$) and simply supported boundary conditions.

N	Ω_1	Relative Change
10	4.375234	—
14	4.375487	5.8×10^{-5}
18	4.375499	2.7×10^{-6}
22	4.375500	2.3×10^{-7}
26	4.375500	$< 10^{-8}$

4.2 Problem 1: Linearly Tapered Beams with Elastic Boundaries

4.2.1 Problem Definition

A linearly tapered beam with taper ratio $\alpha = 1.4$ is considered, so that the stiffness and mass distributions are: $I(z) = I_0[0.4z + 1]^4$, $A(z) = A_0[0.4z + 1]^2$. Ten distinct boundary combinations are analyzed following De Rosa and Auciello [22], as detailed in Table 3. In the table E denotes an elastically restrained end, C clamped, S simply supported, F free, and Pseudo-F, a near-free condition modeled with appropriate spring parameters.

Table 3. Boundary condition combinations used in a linearly tapered beam.

Problem	Left BC	Right BC	Description
B1	C	C	Clamped–Clamped
B2	S	S	Simply–Simply
B3	C	S	Clamped–Simply
B4	E	S	Elastic–Simply
B5	E	E	Elastic–Elastic
B6	E (specific)	Pseudo-F	Special elastic-free
B7	$E (K_T = 1/0.4^3)$	Pseudo-F	Modified elastic-free
B8	$E (K_T = 10/0.4^3)$	Pseudo-F	Enhanced elastic-free

Problem	Left BC	Right BC	Description
B9	$E(W'' = 0, W = 0)$	Pseudo-F	Moment-free, displacement-zero
B10	$E(W'' = 0, W = 0)$	S	Moment-Simply

4.2.2 Complete Frequency Results

Table 4 presents the first five nondimensional frequencies for all ten boundary combinations of Table 3 computed with $N = 25$ collocation points and multi-precision arithmetic.

Table 4. Non-dimensional frequencies Ω_i ($i = 1, \dots, 5$) for a linearly tapered beam ($\alpha = 1.4$)

Problem	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
B1	4.063575	7.761934	11.505381	15.269523	19.043637
B2	3.730038	7.630248	11.421711	15.208334	18.995442
B3	3.798407	7.680343	11.460457	15.239750	19.021795
B4	4.124910	7.810550	11.543621	15.300707	19.069865
B5	2.376612	5.373870	8.726374	12.113521	15.519098
B6	2.442011	5.380548	8.727984	12.114129	15.519388
B7	2.855427	5.441420	8.742585	12.119617	15.522005
B8	4.432887	7.800821	11.206093	14.621945	18.042729
B9	3.415951	6.868668	10.297851	13.725998	17.154266
B10	4.432887	7.800821	11.206093	14.621945	18.042729

4.2.3 Validation Against De Rosa and Auciello [22]

Table 5 compares the present results for Problem B2 (simply supported, $\alpha = 1.4$) with the Bessel function solution of De Rosa and Auciello [22]. Relative errors are below 0.001% for all five modes, and the present method provides two to three additional verified decimal digits.

Table 5. Comparison of nondimensional frequencies for a simply supported linearly tapered beam ($\alpha = 1.4$) between De Rosa and Auciello [22] (Bessel functions) and the present Chebyshev method ($N = 25$).

Mode	De Rosa & Auciello [22]	Present ($N = 25$)	Rel. Error
1	3.7300	3.730038	< 0.001%
2	7.6302	7.630248	< 0.001%
3	11.4217	11.421711	< 0.001%
4	15.2083	15.208334	< 0.001%
5	18.9954	18.995442	< 0.001%

4.3 Problem 2: Exponentially Varying Beams with Damaged Boundaries

4.3.1 Problem Definition

The exponentially varying beam with $\beta = 1.0$ is considered, so that $I(z) = I_0 e^{4z}$ and $A(z) = A_0 e^{2z}$ (strong gradation). The left end ($z = 0$) is clamped, satisfying $W(0) = 0$ and $W'(0) = 0$. The right end ($z = 1$) is elastically restrained with a rotational spring (parameter α_T) and a translational spring (parameter α), imposing:

$$W'(1) = \alpha_T W''(1), \quad W(1) = \alpha[W'''(1) + 4\beta W''(1)] \quad (20)$$

The damage parameters $\alpha_T \in [0,1]$ and $\alpha \in [0,1]$ represent rotational and translational stiffness reductions, respectively.

4.3.2 Frequency Results under Damage Variation

Tables 6 and 7 present the first three nondimensional frequencies as a function of translational damage parameter α for two levels of rotational damage with $N=25$.

Table 6. Nondimensional frequencies Ω_i ($i = 1,2,3$) versus translational damage parameter α for an exponentially varying beam with $\beta = 1.0$ and moderate rotational damage $\alpha_T = 0.5$.

α	Ω_1	Ω_2	Ω_3
0.0	16.213245	43.857795	86.374368
0.2	7.137259	24.595044	56.205003
0.4	7.301378	24.659982	56.237149
0.6	7.354727	24.681652	56.247877
0.8	7.381157	24.692491	56.253243
1.0	7.396937	24.698996	56.256463

Table 7. Nondimensional frequencies Ω_i ($i = 1,2,3$) versus translational damage parameter α for an exponentially varying beam with $\beta = 1.0$ and severe rotational damage $\alpha_T = 0.8$.

α	Ω_1	Ω_2	Ω_3
0.0	16.040484	43.592677	86.051203
0.2	7.019223	24.279812	55.736303
0.4	7.192327	24.347742	55.769477
0.6	7.248508	24.370414	55.780549
0.8	7.276324	24.381755	55.786087
1.0	7.292927	24.388561	55.789411

4.3.3 Validation Against Sari and Butcher [35]

Table 8 compares the present results with the independent Chebyshev polynomial solution of Sari and Butcher [35] for $\beta = 1.0$, $\alpha_T = 0.5$, and $\alpha = 0.0$. Relative errors are below 0.002% for all three modes with $N=25$

Table 8. Comparison of nondimensional frequencies for an exponentially varying beam ($\beta = 1.0$, $\alpha_T = 0.5$, $\alpha = 0.0$)

Source	Ω_1	Ω_2	Ω_3
Sari & Butcher [35]	16.213	43.858	86.374
Present ($N = 25$)	16.213245	43.857795	86.374368
Relative Error	0.0015%	0.0005%	0.0004%

4.4 Problem 3: Functionally Graded Beams with Various Boundary Conditions

4.4.1 Problem Definition

A power-law FGM beam with effective modulus and density given by Eqs. (7) is analyzed for four boundary configurations: simply supported at both ends (SS), clamped at both ends (CC), clamped-free cantilever (CF),

and simply supported–clamped (SC), following the parameterization of Wattanasakulpong and Ungbhakorn [42].

4.4.2 Results for SS, CC, CF, and SC Boundaries

Tables 9–12 present the first five nondimensional frequencies for five power-law indices $n \in \{0.2, 0.5, 1.0, 2.0, 5.0\}$ for each boundary configuration and $N=25$

Table 9. First five nondimensional frequencies Ω_i for a power-law FGM beam with simply supported (SS) boundary conditions.

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.2	5.101909	20.407638	45.917185	81.630551	127.547719
0.5	4.668964	18.675856	42.020676	74.703425	116.724086
1.0	4.220986	16.883945	37.988876	67.535780	105.524642
2.0	3.851830	15.407319	34.666468	61.629277	96.295733
5.0	3.667478	14.669912	33.007303	58.679649	91.686940

Table 10. First five nondimensional frequencies Ω_i for a power-law FGM beam with clamped–clamped (CC) boundary conditions.

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.2	11.565456	31.880625	62.498772	103.313645	154.332740
0.5	10.584017	29.175252	57.195159	94.546503	141.236144
1.0	9.568502	26.375944	51.707397	85.474954	127.684817
2.0	8.731666	24.069173	47.185203	77.999538	116.517837
5.0	8.313761	22.917203	44.926881	74.266416	110.941197

Table 11. First five nondimensional frequencies Ω_i for a power-law FGM beam with clamped–free (CF) boundary conditions.

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.2	1.817539	11.390323	31.893234	62.498009	103.313693
0.5	1.663304	10.423746	29.186791	57.194461	94.546547
1.0	1.503713	9.423608	26.386376	51.706766	85.474994
2.0	1.372202	8.599444	24.078693	47.184627	77.999574
5.0	1.306527	8.187868	22.926267	44.926332	74.266450

Table 12. First five nondimensional frequencies Ω_i for a power-law FGM beam with simply supported–clamped (SC) boundary conditions.

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.2	7.970156	25.828411	53.888918	92.153239	140.621371
0.5	7.293813	23.636626	49.315933	84.333163	128.688315
1.0	6.593986	21.368739	44.584167	76.241564	116.340928
2.0	6.017293	19.499884	40.684952	69.573676	106.166054
5.0	5.729301	18.566604	38.737737	66.243823	101.084859

4.4.3 Validation Against Wattanasakulpong and Ungbhakorn [42]

Table 13 compares the present results for the SS boundary with $n = 1.0$ against the DTM results of Wattanasakulpong and Ungbhakorn [42]. Relative errors of 0.0003% are achieved for all three reported modes.

Table 13. Comparison of nondimensional frequencies for a power-law FGM beam ($n = 1.0$, SS boundary)

Source	Ω_1	Ω_2	Ω_3
Wattanasakulpong & Ungbhakorn [42] (DTM)	4.2210	16.8839	37.9889
Present ($N = 25$)	4.220986	16.883945	37.988876
Relative Error	0.0003%	0.0003%	0.0003%

5. Implementation Details

5.1 Computational Parameters

The method uses Chebyshev–Gauss–Lobatto collocation with $N = 25$ points (unless otherwise stated) on the Chebyshev domain $\xi \in [-1,1]$, mapped from the physical domain $z \in [0,1]$ via $\xi = 2z - 1$. Numerical precision is 34-digit multiprecision arithmetic implemented through the MATLAB Symbolic Math Toolbox. Eigenvalues are obtained using the `eig()` function applied to multiprecision matrices, and convergence is assessed by a relative change criterion of $< 10^{-8}$ between successive values of N . The governing equation is enforced at interior points $j = 1, 2, \dots, N - 1$, while boundary conditions are enforced at $j = 0$ and $j = N$ using the analytical derivative formulas of Section 3.2.

5.2 Convergence and Stability Metrics

Convergence is assessed by computing the relative error $\epsilon = \frac{|\Omega_N - \Omega_{N-4}|}{\Omega_N}$ across the sequence $N \in \{10, 14, 18, 22, 26\}$. Comparison with literature is quantified by $\epsilon = \frac{|\Omega_{\text{present}} - \Omega_{\text{reference}}|}{\Omega_{\text{reference}}} \times 100\%$

6. Conclusions

This study has presented a comprehensive Chebyshev collocation framework for free vibration analysis of beams with variable cross-sections and elastic boundary conditions. The key findings are as follows.

The use of exact analytical derivative formulas at boundary nodes — particularly the corrected fourth-derivative formula — eliminates numerical differentiation errors and provides spectral accuracy with only 20–25 collocation points. The variable stiffness boundary conditions properly account for property variation at beam ends. The FGM properties are correctly described through a through-thickness coordinate ζ distinct from the axial coordinate z , with the governing equation parametrized by the effective quantities EI_{eff} and $(\rho A)_{\text{eff}}$ obtained by cross-sectional integration. The combination of multiprecision arithmetic and the CGL point distribution ensures reliable separation of closely spaced eigenvalues.

The present method agrees with three independent literature sources to within: errors below 0.001% compared to De Rosa and Auciello [22] (Bessel functions), errors below 0.002% compared to Sari and Butcher [35] (Chebyshev polynomials), and errors below 0.0005% compared to Wattanasakulpong and Ungbhakorn [42] (DTM). Benchmark-quality data with six-decimal precision are provided for ten boundary combinations for tapered beams ($\alpha = 1.4$), a complete damage parameter sweep for exponential beams ($\beta = 1.0$), and five power-law indices across four boundary types for FGM beams.

The proposed framework demonstrates good computational efficiency and accuracy, making it particularly well suited for parametric studies requiring large numbers of frequency evaluations, structural health monitoring and damage detection algorithms, optimization of tapered and FGM structural components, and validation of commercial finite element software.

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