Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 47 No. 1 (2026)

A Mult-order Non-fourier Fractional Differential Equation for Heat Transmission

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Abstract :- A multi-order fractional differential model was proposed with two fractional Cattaneo derivatives with respect to time and a Caputo symmetrized fractional temperature gradient with respect to space. A unique solution in the form of integral was developed for the proposed heat transfer model using contour integral methods. Numerical examples in graphical forms are provided using the solutions established. The effects of varying parameters of the model are studied from the numerical examples given.

Keywords: *Fractional Caputo derivative, Fourier transform, Laplace transform, Contour integral.*

1. Introduction

The so-called Fourier heat transmission differential equation which is obtained from the combination of the constitutive Fourier law (2) and energy balance equation (3) produces an unlimited speed of heat transmission due to its parabolic nature [1, 2]. There are myriads of natural phenomena that the model cannot adequately describe. To take into many problems which occur natural with finite speed, a relaxation time was introduced to the Fourier law (2) to produce a constitutive Cattaneo heat model equation (1)

$$q(x,t) + \tau\partial_t q(x,t) = -\lambda\partial_x T(x,t) \tag{1}$$

$$q(x,t) = -\lambda\partial_x T(x,t) \tag{2}$$

$$\rho c\partial_t T(x,t) = -\lambda\partial_x q(x,t) \tag{3}$$

where from equation (1,2,3) ρ is density of medium, τ represents relaxation time, c is specific heat capacity, $q(x,t)$ is heat flux, $T(x,t)$ denotes temperature of medium and λ describe the thermal conductivity of the medium. The telegraph equation (4) which is an example of hyperbolic equation can be generated by combining (1) and (3).

$$\partial_t^2 T(x,t) + \tau\partial_t T(x,t) = -\lambda\partial_x^2 T(x,t) \tag{4}$$

Equations such as (1) and (4) which are hyperbolic differential equations facilitate a fixed speed of heat transmission in a medium ([2],[1]).

Experiments conducted using the classical Cattaneo-type and Jeffry-type hyperbolic differential model equations in certain porous materials using laser pulse heating suggest that there are still discrepancies between experimentally produced data and theoretical predictions of the models ([1]). Successful application of fractional calculus in solving many practical problems in the field of science and non-science domains is due to their global dependency or nonlocal property [3–5,1]. The one-dimensional multi-order fractional derivative heat transmission equation is modeled by the system

$$\partial_t T(x,t) = -\partial_x \bar{q}(x,t) \tag{5}$$

$${}_0^C\partial_t^{\beta-1}\bar{q}(x,t) + \tau\,{}_0^C\partial_t^{\alpha-1}\bar{q}(x,t) = \mathfrak{D}\,{}_a^C\varepsilon_b^{\vartheta} T(x,t) \tag{6}$$

where for the fractional scalar orders of equation (6), $0 < \beta \leq \alpha \leq 2$, $0 < \vartheta < 1$, $\bar{q} = \frac{q}{\rho C}$, and $\mathfrak{D} = \frac{\lambda}{\rho C}$ is the thermal diffusivity. The differential equation is subject to the following assumptions:

1. the temperature on any point along the one-dimensional medium at time $t = 0$ is fixed at a value of T_0.
2. heat flux on any point along the one-dimensional medium, $q(x,t)$ at time $t = 0$ is zero.

3. temperature at the two extreme ends of the one- dimensional medium is zero at any time, t as one approaches any of the two infinite endpoints $(-\infty, +\infty)$.
4. heat flux at the two extreme ends of the one- dimensional medium is zero at any time, t as one approaches any of the two infinite endpoints $(-\infty, +\infty)$.

This differential model can be applied to both finite and infinite length media depending on how the boundary points are fixed. By combining equation (5) and (6) of the model, a single model differential equation is generated.

$$ {}_0^C\partial_t^{\beta} T(x,t) + \tau\, {}_0^C\partial_t^{\alpha} T(x,t) = \mathfrak{D}\partial_x\left({}_a^C\varepsilon_b^{\vartheta} T(x,t) \right) \quad (7)$$

Some limiting cases of the model (7) are listed as follows: When $\vartheta = 1$ in equation (6) and (7), the fractional Cattaneo heat conduction models studied in the articles [1] and [6] are obtained respectively. For $1 < \alpha < 2$ and $\beta = 2$ in equation (6), the fractional Cattaneo heat conduction equation studied in [2] is obtained. From equation (7), the telegraph equation is derived (i.e when $\alpha = 2, \beta = 1, \vartheta = 1$), the heat diffusion equation when $\tau = 0$, $\beta = 1, \vartheta = 1$ and lastly the wave equation when $\vartheta = 1$, $\beta = 2$, $\alpha = 2$. This work is arranged with the current section 1 dealing with the introductory aspects of the differential problem stated in (7) and how the rest of the study is formatted in the remaining sections. Section 2 contains propositions, notations needed to enhance understanding of the mathematical model problem and the space domain that is considered ideal for deploying the tools relevant in solving the problem. Section 3 provides the requisite theorems necessary in establishing a

distinct solution which exists within the boundary conditions of the model. Section 4 provides an exact solution using contour integration. Numerical examples are provided using the explicit solution obtained.

2. Mathematical Preliminaries

In this section, the operating space of Schwartz tempered distributions in the $\mathbb{R}^n$is denoted by $S'(\mathbb{R}^n)$. The subspaces $S'_+(\mathbb{R}^n)$ and $S'_-(\mathbb{R}^n)$ of the Schwartz space $S'(\mathbb{R})$ are defined in the sets $[0, \infty)$ and $[-\infty, 0)$ respectively.

For $0 \le \alpha < 1$ and $-\infty \le a \le b \le \infty \in \mathbb{R}$, the left Caputo derivative and right Caputo derivative of a function $u: [a, b] \to \mathbb{R}$ are defined as:

$$ {}_{a^+}^C D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^{\alpha}} ds $$

$$ {}_t^C D_{b^-}^{\alpha} u(t) = -\frac{1}{\Gamma(1-\alpha)} \int_b^t \frac{u'(s)}{(t-s)^{\alpha}} ds $$

In [2], the fractional Caputo derivative with respect to space, x of order $\vartheta \in [0, 1)$ of definitely continuous function u is defined as

$$ {}_a^C\varepsilon_b^{\vartheta} u(x) = \frac{1}{2}\left({}_a^C D_x^{\vartheta} - {}_x^C D_b^{\vartheta} \right) u(x) $$

$$ = \frac{1}{2} \frac{1}{\Gamma(1-\vartheta)} \int_a^b \frac{u'(\theta)}{|x-\theta|^{\vartheta}} d\theta \quad (8)$$

For simplicity and with $a = -\infty$ and $b = \infty$, ${}_a^C\varepsilon_b^{\vartheta} u(x)$ is written as $\varepsilon_x^{\vartheta} u(x)$.

$$ \varepsilon_x^{\vartheta} u(x) = \frac{1}{2} \frac{1}{\Gamma(1-\vartheta)} |x|^{-\vartheta} * u'(x) = \sin \frac{\pi\vartheta}{2} \frac{d}{dx} I^{1-\vartheta} u(x) $$

Where I^{ϑ} is the Riesz potential. To deal with fractional operators in the distributional space, a family of operators $(J_\alpha)_\alpha \in S'_+$ is introduced as

$$ J_\alpha(t) = \begin{cases} H(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0 \\ \frac{d^n}{dt^n} J_{\alpha+n}(t), & \alpha \le 0, \quad \alpha + n > 0, \quad n \in \mathbb{N} \end{cases} $$

and $\left(\check{J}_\alpha\right)_\alpha \in S'_-(\mathbb{R}^n)$ as $\check{J}_\alpha(t) = J_\alpha(-t)$.

Where H represents a Heaviside function. For $\alpha < 0$, $\check{J}_\alpha *$ and $J_\alpha *$ respectively denote convolution of left and right sided fractional differentiation of a function u which is continuous, hence

${}_{a^+}^{C}D_t^{\alpha}u = J_{1-\alpha} * u'$ and ${}_{t}^{C}D_{b^-}^{\alpha}u = -\check{J}_{\alpha} * u'$

By choosing $v_1, v_2, v_3 \in S'(\mathbb{R})$ and $\psi \in S(\mathbb{R})$, Fourier and Laplace transforms including some of their properties are defined.

Fourier transform, $\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) := \int_{-\infty}^{\infty} \psi(x)e^{-i\xi x}\,dx,\ \xi \in \mathbb{R}$

$\mathcal{F}(v_1 * v_2)(\xi) = \mathcal{F}v_1(\xi) x \mathcal{F}v_2(\xi),\ \mathcal{F}\left[\frac{d^n v}{dx^n}\right] = (i\xi)^n\, \mathcal{F}v(\xi), n \in \mathbb{N},\ \mathcal{F}\delta(\xi) = 1$

$\langle \mathcal{F}v, \psi \rangle = \langle \psi, \mathcal{F}v \rangle$

Fourier inverse transform, $\mathcal{F}^{-1}\big(v(\xi)\big)x := \frac{1}{2\pi}\int_{-\infty}^{\infty} v(\xi)e^{i\xi x}\,d\xi$

Laplace transform and its properties:

Laplace transform, $\mathcal{L}v(s) := \int_0^{\infty} v(t)e^{-st}\,dt$

$\mathcal{L}(v_1 * v_2)(s) = \mathcal{L}v_1(s) x \mathcal{L}v_2(s) \quad \mathcal{L}\left[{}_{0}^{C}D_t^{\alpha}v\right](s) = s^{\alpha}\mathcal{L}v(s), \alpha \geq 0 \quad \mathcal{L}\delta(s) = 1$

Inverse Laplace transform, $\mathcal{L}^{-1}v(s) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} v(s)e^{st}\,ds,\ t > 0$

3. Establishing a Unique Solution that Exist for the System Equations (5), (6).

The aim of this part of the study is to establish a solution to the system (5) and (6). To start the section, some propositions and remarks are stated.

Proposition 1 $\big(T(x,t), q(x,t)\big) \in E^2$ is a solution that is generalized for the systems (5), (6) and satisfy the initial conditions only if

$$\partial_t T(x,t) = -\partial_x \overline{q}(x,t) + T_0(x)\delta(t) \tag{9}$$

$${}_{0}^{C}\partial_t^{\beta-1} q(x,t) + \tau\, {}_{0}^{C}\partial_t^{\alpha-1} q(x,t) = -\lambda \varepsilon_x^{\vartheta} T(x,t) \tag{10}$$

Holds in E.

Remarks 1: (i) if $\big(T(x,t), q(x,t)\big) \in E^2$ as in Proposition 1, then the boundary conditions of the system (5), (6) are equally satisfied, since $T, q \in E$, implies the boundary conditions are confined within solution domain.

(ii) suppose $\big(T(x,t), q(x,t)\big) \in E^2$ is a solution that is generalized for (5), (6) including the initial conditions, then $T(x,t)$ is a generalized solution to the equivalent form of (9) and (10), i.e

$$\partial_t T = \mathfrak{D}\mathcal{L}^{-1}\left(\frac{s}{\tau s^{\alpha} + s^{\beta}}\right)\partial_x \varepsilon_x^{\vartheta} T + T_0(x)\delta(t) \tag{11}$$

is valid in E.

Main Theorem Let $T_0 \in E$ and $x \in \mathbb{R}$, then a solution $T(x,t) \in E^2$ which is unique to (11) exist and is defined by

$$T(x,t) = \frac{T_0 * m_0}{2\pi i}\int_0^{\infty}\Bigg(\left(\tau q^{\alpha}e^{i\alpha\pi} + q^{\beta}e^{i\beta\pi}\right)^c e^{ib_0\left(\frac{\tau q^{\alpha}e^{i\alpha\pi} + q^{\beta}e^{i\beta\pi}}{\Omega}\right)^c |x|}$$

$$-\left(\tau q^{\alpha}e^{-i\alpha\pi} + q^{\beta}e^{-i\beta\pi}\right)^c e^{ib_0\left(\frac{\tau q^{\alpha}e^{-i\alpha\pi} + q^{\beta}e^{-i\beta\pi}}{\Omega}\right)^c |x|}\Bigg)\frac{e^{-qt}}{q}\,dq$$

Where $c = \frac{1}{1+\vartheta}$, $\Omega = \lambda\sin\left(\frac{\pi\vartheta}{2}\right)$, $b_0 = e^{\frac{i\pi}{1+\vartheta}}$, $m_0 = \frac{1}{2\sin\left(\frac{\pi}{1+\vartheta}\right)}\left(\frac{1}{\Omega}\right)^c$

To proof the **main theorem**, a lemma is stated as follows:

[1]***lemma***: Consider the functions $u, u_0 \in S'(\mathbb{R})$ in Laplace transform state, with

$f(s), w(s) \in \mathbb{C},\quad \omega(s) \in \mathbb{C}\backslash(-\infty, 0]$. Then

$$\partial_x \varepsilon_x^{\vartheta} \tilde{u} - w(s)\tilde{u} = -f(s)\widetilde{u_0} \quad (12)$$

has a unique solution in Fourier-Laplace transform domain given by

$$\hat{\tilde{u}} = \frac{f(s)\widehat{\widetilde{u_0}}}{\sin\left(\frac{\pi\vartheta}{2}\right)|\xi|^{1+\vartheta} + w(s)} \quad (13)$$

To obtain (13), first take the Laplace transform of the system (9) and (10) as

follows:

$$s\tilde{T} = -\partial_x \tilde{q} + \tilde{T}_0(x) \quad (14)$$

$$s^{\beta-1}\tilde{q} + \tau s^{\alpha-1}\tilde{q} = -\lambda \varepsilon_x^{\vartheta} \tilde{T} \quad (15)$$

For simplification of the model (with $\rho c = 1$) in (5) and (6), $\mathfrak{D} = \frac{\lambda}{\rho c} \to \lambda$, $\bar{q} = \frac{q}{\rho c} \to q$. Substituting (15) into (14) yields

$$\partial_x\left(\varepsilon_x^{\vartheta}\tilde{T}\right) - \frac{s(\tau s^{\alpha-1} + s^{\beta-1})}{\lambda}\tilde{T} = -\tilde{T}_0(x)\frac{(\tau s^{\alpha-1} + s^{\beta-1})}{\lambda} \quad (16)$$

By setting $w(s) := \frac{s(\tau s^{\alpha-1} + s^{\beta-1})}{\lambda}$ and $f(s) := \frac{(\tau s^{\alpha-1} + s^{\beta-1})}{\lambda}$, and taking the Fourier transform, equation (16) has a solution of the form as in (13).

With $^{1}\textit{lemma}$ a unique solution (13) is guaranteed for the model system. A second lemma is considered to verify that the unique solution (13) exist.

$^{2}\textit{lemma}$: If $\tau, \Omega > 0$ and $0 < \beta \le \alpha \le 2 (0 < \beta \le 1, 1 \le \alpha \le 2)$, then there exist a unique $r_0 > 0$ and $\psi_0 \in \left(\frac{\pi}{\alpha}, \frac{\pi}{\alpha-\beta}\right)$, such that a unique solution $s_0 = r_0 e^{i\psi_0}$ and its complex conjugate $\bar{s}_0$ are zeros of the function, $z(s)$.

$$z(s) = \tau s^{\alpha} + s^{\beta} + \Omega$$

Where $\Omega = \sin\left(\frac{\pi\vartheta}{2}\right)|\xi|^{1+\vartheta}$ and $z(s)$ is the denominator of equation (13) when $f(s)$ and $w(s)$ are substituted.

Proof of $^{2}\textit{lemma}$

By considering a solution of the form $s = re^{i\psi}$ and splitting $z(s) = 0$ into its real and imaginary parts, the following equations are obtained

$$\tau r^{\alpha}\cos(\alpha\psi) + r^{\beta}\cos(\beta\psi) + \Omega = 0 \quad (17)$$

$$\tau r^{\alpha}\sin(\alpha\psi) + r^{\beta}\sin(\beta\psi) = 0 \quad (18)$$

The complex solution $s_0 = r_0 e^{i\psi_0}$ and its conjugate solution $\bar{s}_0 = r_0 e^{-i\psi_0}$ both satisfy (17), (18). This implies that a change in argument from positive to negative does not affect the solution. As a result, the focus is on the upper half complex plane only. For $\psi = 0$, equation (17) cannot be considered true as the left-hand side will always be greater than zero since $\tau, \Omega > 0$. For $\psi = \pi$, equation (18) will not be valid as $\sin(\beta\pi) > 0$. For $\psi \in \left(0, \frac{\pi}{2}\right]$, equation (18) cannot be satisfied since all terms will be positive. If a solution exists then it must be in $\psi \in \left(\frac{\pi}{2}, \pi\right)$. If a solution exists, its real part must be negative. we therefore consider $\psi \in \left(\frac{\pi}{\alpha}, \pi\right)$ where $\sin(\beta\psi) > 0$ and $\sin(\alpha\psi) < 0$ for a solution to equation (18). For $r^{\beta} \neq 0$ in (18)

$$r = \left(\frac{\sin(\beta\psi)}{-\tau\sin(\alpha\psi)}\right)^{\frac{1}{\alpha-\beta}}$$

Substituting r into equation (17), yields

$$g(\psi) := \left(\frac{(\sin(\beta\psi))^{\beta}}{(\sin(\alpha\psi))^{\alpha}}\right)^{\frac{1}{\alpha-\beta}} \sin((\alpha-\beta)\psi) = \Omega\tau^{\frac{\beta}{\alpha-\beta}} \quad (19)$$

The requirement that $\sin((\alpha-\beta)\psi)>0$ is upheld since $\Omega\tau^{\frac{\beta}{\alpha-\beta}}>0$ for equation (19). The arguments of the solutions on the upper half complex plane are further

restricted to the interval (see ([7]))

$\frac{\pi}{\alpha}<\psi<min\left(\pi,\frac{\pi}{\alpha-\beta}\right)$.

$g(\psi)$ is continuous and positive on the interval $\left(\frac{\pi}{\alpha},\frac{\pi}{\alpha-\beta}\right)$ with limits

$\lim_{\psi\to\frac{\pi}{\alpha}^{+}} g(\psi)=\infty$ and $\lim_{\psi\to\frac{\pi}{\alpha-\beta}^{-}} g(\psi)=0$

This implies that there exist a ψ_0 such that $g(\psi_0)=\Omega\tau^{\frac{\beta}{\alpha-\beta}}$. Therefore, a solution to (17), (18) exist and is given by $s_0=r_0e^{i\psi_0}$, where $r_0=\left(\frac{(sin(\beta\psi_0))^{\beta}}{(sin(\alpha\psi_0))^{\alpha}}\right)^{\frac{1}{\alpha-\beta}}$. The function $g(\psi)$ is monotonically decreasing if $\frac{dg(\psi)}{d\psi}<0.$

Proof:

$$\frac{dg(\psi)}{d\psi}=-\frac{\sin(\beta\psi)^{\frac{\beta}{\alpha-\beta}-1}}{(\alpha-\beta)(-\sin(\alpha\psi))^{\frac{\beta}{\alpha-\beta}+1}}m_0(\psi)$$

with $m_0(\psi):=\alpha^2sin^2(\beta\psi)+\beta^2sin^2(\alpha\psi)-2\alpha\beta\sin(\alpha\psi)\sin(\beta\psi)\cos((\alpha-\beta)\psi)>0$.

For $\psi\in\left(\frac{\pi}{\alpha},\frac{\pi}{\alpha-\beta}\right)$, $\sin(\beta\psi)>0$, $\sin(\alpha\psi)<0$, $\cos((\alpha-\beta)\psi)\le 1$. This reduces m_0 to a perfect square as

$m_0=(\beta\sin(\alpha\psi)-\alpha\sin(\beta\psi))^2>0$

Evidently, $m_0(\psi)>0$ proves that $\frac{dg(\psi)}{d\psi}<0$. Since there exist a unique $\psi_0\in\left(\frac{\pi}{\alpha},\frac{\pi}{\alpha-\beta}\right)$ such that $g(\psi_0)=\Omega\tau^{\frac{\beta}{\alpha-\beta}}$ and g(ψ) is strictly monotonically decreasing function, s_0 is the unique solution to (17) and (18). We used the approach adopted by ([8][9]) to examine existence and uniqueness of solutions $(s_0,\bar{s}_0)$. An alternative approach in examining existence of solutions (roots) exist (see proposition 2 in [7]).

Proof of Main Theorem

To establish the unique solution as stated in the main theorem, determine the Fourier inverse transform of (13) as obtained from [1]***lemma***.

$$\mathcal{F}^{-1}\left(\frac{1}{\sin\left(\frac{\pi\vartheta}{2}\right)|x|^{1+\vartheta}+w(s)}\right)(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x}d\xi}{sin\left(\frac{\pi\vartheta}{2}\right)|x|^{1+\vartheta}+w(s)}.$$

With the Cauchy residue theorem

$$\frac{1}{2\pi i}\int_{\Gamma}\widehat{\widehat{u}}(\xi,s)e^{i\xi x}\,d\xi=\sum_{k=1}^{n}Res\left(\widehat{\widehat{u}}(\xi_k)\right)\tag{20}$$

where Γ is the Hankel contour integration path. For calculations involving contour integrals see ([7],[8],[10]). The residues for $x\ge 0$, $w(s)\in\mathbb{C}\backslash(-\infty,0]$ from the contour path Γ using fig.1 is calculated as follows

$$\int_{\varepsilon}^{r}\widehat{\widehat{u}}(\xi,s)e^{i\xi x}d\xi+\int_{C_r}\widehat{\widehat{u}}(\xi,s)e^{i\xi x}d\xi+\int_{\gamma_\varepsilon}\widehat{\widehat{u}}(\xi,s)e^{i\xi x}d\xi+\int_{r}^{\varepsilon}\widehat{\widehat{u}}(\xi,s)e^{i\xi x}d\xi=2\pi i\sum_{k=1}^{n}Res\left(\widehat{\widehat{u}}(\xi_k)\right)\tag{21}$$

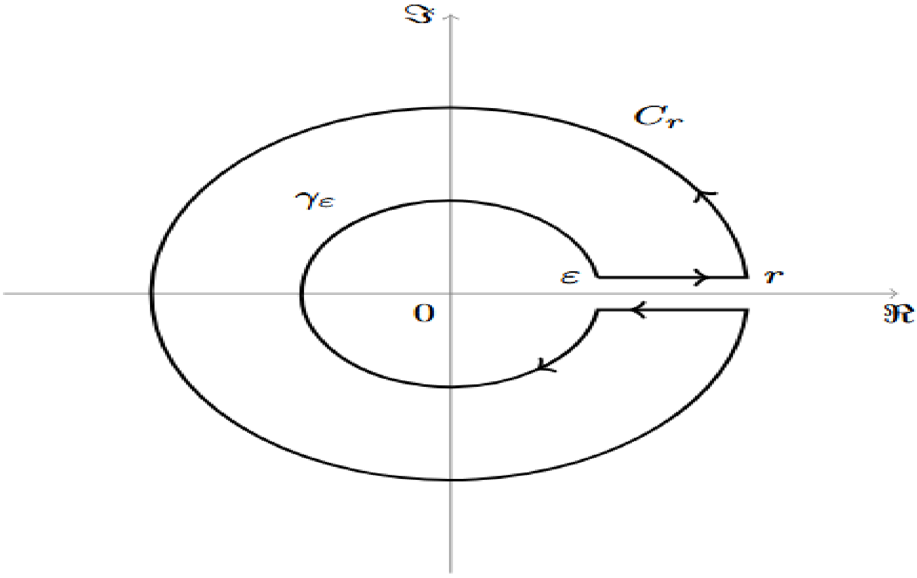

Fig. 1. Contour integration path

Solving for the residue in (21), the Fourier inverse solution of (13) is given as

$$\tilde{u}(x,s) = u_0 * m_0(\tau s^\alpha + s^\beta)^c \frac{1}{s} e^{ib_0\left(\frac{\tau s^\alpha + s^\beta}{\Omega}\right)^c x}, x \geq 0 \tag{22}$$

where m_0, b_0, c, Ω are explained in the **main theorem.**

Considering $-\infty < x < 0$, the poles in (13) will be negative and the corresponding solution will be given as

$$\tilde{u}(x,s) = u_0 * m_0(\tau s^\alpha + s^\beta)^c \frac{1}{s} e^{-ib_0\left(\frac{\tau s^\alpha + s^\beta}{\Omega}\right)^c x} \tag{23}$$

This implies for any $x \in \mathbb{R}$, the solution in the Laplace transform domain is given by

$$\tilde{u}(x,s) = u_0 * m_0(\tau s^\alpha + s^\beta)^c \frac{1}{s} e^{-ib_0\left(\frac{\tau s^\alpha + s^\beta}{\Omega}\right)^c |x|} \tag{24}$$

$\tilde{u}(x,s)$ is a multivalued function with no singularities. The most obvious branch point is located at $s = 0$. Since $\tilde{u}(x,s)$ does not have singularities, its Laplace transform inverse can be calculated with Cauchy integral formula for a closed curve as

$$\oint_{\Gamma_{\gamma,}} \tilde{u}\,(x,s)e^{st}ds = 0 \tag{25}$$

where the path of integration over the close contour in fig. 2 is given by

$$\oint_{\Gamma_\gamma} = \int_{HKA} + \int_{AB} + \int_{BCD} + \int_{DE} + \int_{EFG} + \int_{GH}$$

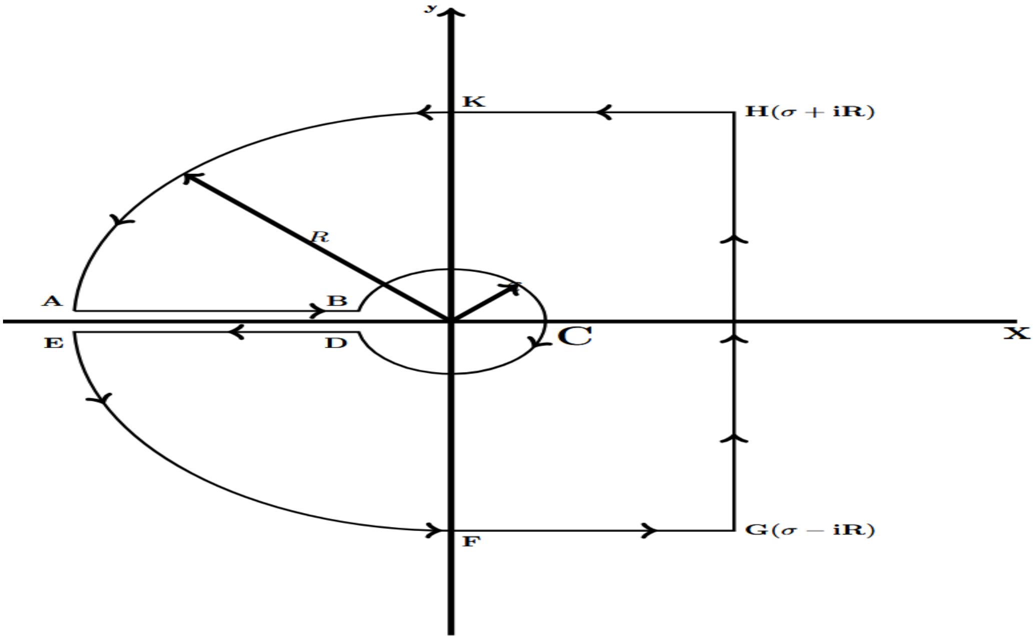

Fig. 2. Bromwich contour path

With Jordan's Lemma, contributions from $\int_{HKA}$ and $\int_{EFG}$ $\boldsymbol{reduces\ to}$ zero as $\boldsymbol{R \to 0}$. The solution after inverting $\boldsymbol{\tilde{u}(x,s)}$ using Laplace transform is

$$\boldsymbol{u(x,t) = \int_{GH} \tilde{u}\,(x,s)e^{st}ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{u}\,(x,s)e^{st}ds} \tag{26}$$

From fig. 2, if $\boldsymbol{s = qe^{i\pi}}$ is along the path AB and $\boldsymbol{s = qe^{-i\pi}}$ along DE, then

$$\boldsymbol{2\pi iu(x,t) = -\int_{AB} \tilde{u}\,(x,s)e^{st}ds|_{s=qe^{i\pi}} - \int_{DE} \tilde{u}\,(x,s)e^{st}ds|_{s=qe^{-i\pi}} - \int_{BCD} \tilde{u}\,(x,s)e^{st}ds|_{s=\varepsilon e^{i\psi}}} \tag{27}$$

$$\boldsymbol{\lim_{\varepsilon\to 0}\int_{BCD} \tilde{u}\,(x,s)e^{st}ds|_{s=\varepsilon e^{i\psi}} = u_0 * m_0 \lim_{\varepsilon\to 0}\int_{\pi}^{-\pi}\left(\tau\varepsilon^{\alpha}e^{i\alpha\psi} + \varepsilon^{\beta}e^{i\beta\psi}\right)^{c} e^{-ib_0\left(\frac{\tau\varepsilon^{\alpha}e^{i\alpha\psi}+\varepsilon^{\beta}e^{i\beta\psi}}{\Omega}\right)^{c}|x|} id\psi = 0} \tag{28}$$

By calculating the remaining integrals in (27), the unique solution as stated in the main theorem is established as

$$\boldsymbol{2\pi iu(x,t) =}$$

$$\boldsymbol{u_0 * m_0\int_{0}^{\infty}\Bigg(\left(\tau q^{\alpha}e^{i\alpha\pi} + q^{\beta}e^{i\beta\pi}\right)^{c} e^{ib_0\left(\frac{\tau q^{\alpha}e^{i\alpha\pi}+q^{\beta}e^{i\beta\pi}}{\Omega}\right)^{c}|x|} - \left(\tau q^{\alpha}e^{-i\alpha\pi} + \right.}$$

$$\boldsymbol{\left. q^{\beta}e^{-i\beta\pi}\right)^{c} e^{ib_0\left(\frac{\tau q^{\alpha}e^{-i\alpha\pi}+q^{\beta}e^{-i\beta\pi}}{\Omega}\right)^{c}|x|}\Bigg)\frac{e^{-qt}}{q}dq} \tag{29}$$

4. Numerical Examples

In all numerical examples, only the unique solution, $\boldsymbol{u(x,t)}$ valid for $\boldsymbol{x \geq 0}$ is considered. Fixed values of the constants $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$, $\boldsymbol{\alpha}$, $\boldsymbol{\Omega}$, $\boldsymbol{\beta}$ were used in the examples. Values of $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ are arbitrarily fixed at $\boldsymbol{\tau = 0.1}$ and $\boldsymbol{\lambda = 1}$.

Example 1 In this example, fixed $\{\boldsymbol{\alpha = 1.9, \beta = 0.5, t = 1.0}\}$ for figure 3(a), and $\{\boldsymbol{\alpha = 1.8,\ \beta = 0.7,\ t = 1.0}\}$ for figure 3(b) are considered. A plot of temperature along different spatial coordinates $\boldsymbol{x}$ for different values of $\boldsymbol{\vartheta} = \{\boldsymbol{0.6, 0.7, 0.8, 0.9}\}$ is shown in figure 3(a). In figure 3(b) temperature prediction is shown for increasing values of $\boldsymbol{\vartheta} = \{\boldsymbol{0.6, 0.7, 0.8, 0.9, 1.0}\}$. Higher and sharper peaks temperature values are observed in both figure 3(a and b) as $\boldsymbol{\vartheta}$ decreases.

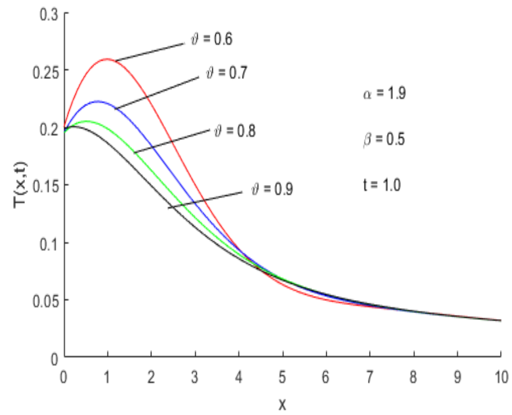

Fig. 3(a)

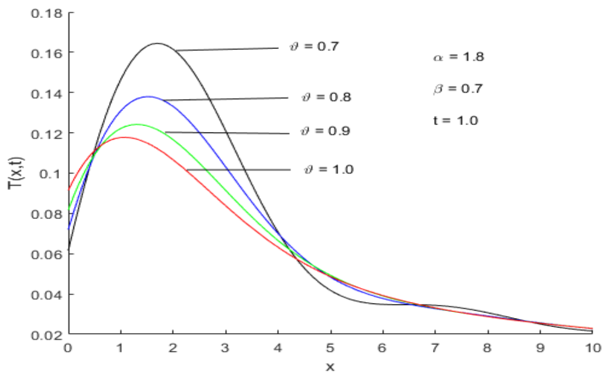

Fig. 3(b)

Example 2: I fixed $\boldsymbol{\alpha = 1.8}$, $\boldsymbol{\vartheta = 0.9}$ and plot $\boldsymbol{T(x,t)}$ variations along spatial coordinate points x at different $\boldsymbol{\beta} \in \{\mathbf{0.6, 0.7, 0.8, 0.9, 1.0}\}$ at different time instant $\boldsymbol{t} \in \{\mathbf{0.3, 1.0, 1.5, 2.0, 2.5}\}$ as shown in fig. 4(a). Fig.4(b) shows plots of $\boldsymbol{T(x,t)}$ versus medium point values $\boldsymbol{x}$ for $\boldsymbol{\alpha} \in \{\mathbf{0.9, 1.2, 1.4, 1.6, 1.8, 2.0}\}$ for different time instants $\boldsymbol{t} \in \{\mathbf{2.0, 2.5, 3.0, 3.6, 4.5, 5.5}\}$. In figure 4(a), the smaller the parameter $\boldsymbol{\beta}$, the higher and narrower the peak temperature values at different time instants, t. Figure 4(b) also shows a similar for trend for the α parameter on temperature distribution along the medium at different time instants, t

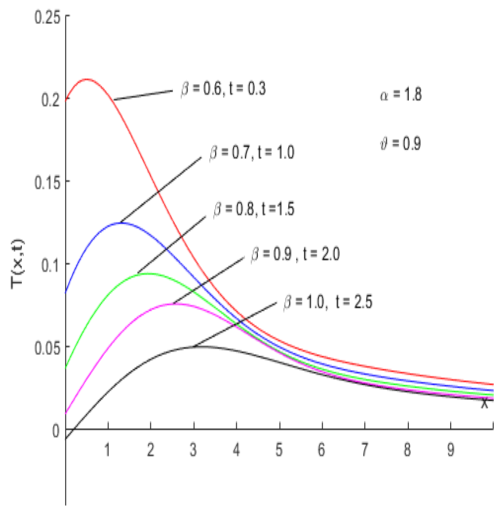

Fig. 4(a)

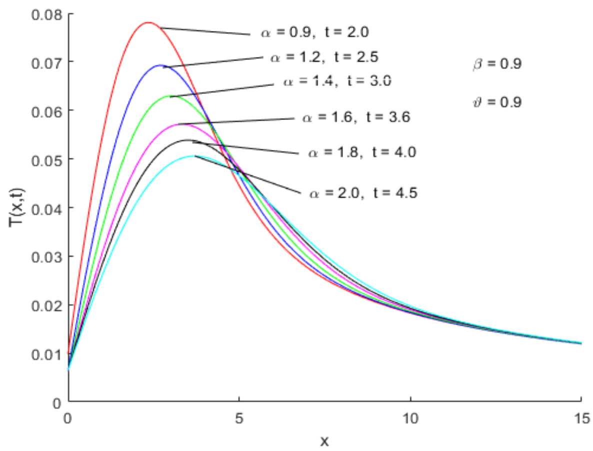

Fig. 4(b)

Example 3 For this example, I fixed $\boldsymbol{\alpha = 1.8}$, $\boldsymbol{\vartheta = 0.4}$ and $\boldsymbol{\beta = 1.0}$ for different spatial points $\boldsymbol{x} \in \{\mathbf{1.0, 1.2, 1.4, 1.6, 1.8, 2.0}\}$ and plot temperature function $\boldsymbol{T(x,t)}$ against time, $\boldsymbol{t} \in (\mathbf{0, 5})$ as shown in figure 5. Sharper, higher and narrower peaks of the Temperature distribution function $\boldsymbol{T(x,t)}$ are observed for smaller spatial points inside the medium than bigger spatial points. The peak temperature values decrease with increase in the medium spatial points. The temperature of the medium after peaking reduces sharply or gradually depending on the location of the spatial point $\boldsymbol{x}$ at which the heating started.

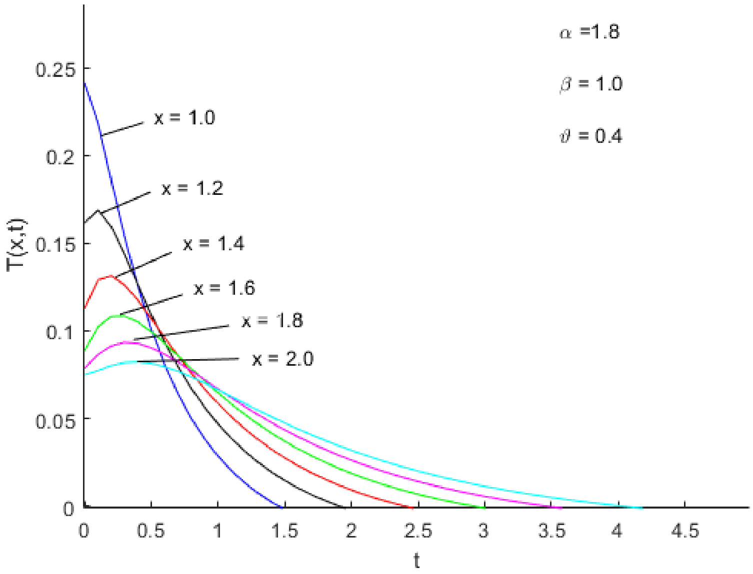

Fig. 5

5. Conclusions

I proposed a new generalization of the Classical Cattaneo constitutive equation by modifying the model used in ([1]). I introduced the symmetrized fractional Cattaneo temperature gradient of order $\boldsymbol{\vartheta}$ with respect to spatial point $\boldsymbol{x}$. The parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are fractional orders of derivatives in respect of time. Our model contains three fractional parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$, where $\mathbf{0} < \boldsymbol{\beta} \leq \boldsymbol{\alpha} \leq \mathbf{2}$ and $\mathbf{0} < \boldsymbol{\vartheta} < \mathbf{1}$. A unique solution to the model (5), (6) was established and its existence proven. Explicit solutions in integral form are provided. Effects of non-integer orders for derivatives of the model are examined using numerical examples provided. Smaller values of β produced sharper and stronger effect than bigger values of $\boldsymbol{\beta}$. As $\boldsymbol{\alpha}$ values increase, the effect become stronger. For the parameter $\boldsymbol{\vartheta}$, smaller values produce stronger effect on temperature distribution in the medium than larger values of $\boldsymbol{\vartheta}$.

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