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# Optimal Inventory Management with Time-Varying Exponential Demand, Weibull Decay, Partial Shortages, and Permissible Payment Delays

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***Abstract***

This study presents a comprehensive inventory model that integrates exponential demand, gradual deterioration following a Weibull distribution, and partial backlogging, under a trade credit policy framework. The primary objective is to develop a price and lot-sizing model suitable for retail environments where suppliers permit a delay in payments—a realistic and common trade credit scenario. The model assumes that demand increases exponentially over time, while items deteriorate gradually rather than instantaneously. Additionally, the possibility of shortages is allowed, with partial backlogging occurring only during the shortage period for the remaining unmet demand. To determine the optimal ordering policy, the first and second order conditions are established, leading to the development of a structured algorithm aimed at minimizing the total cost. The algebraic approach, combined with cost minimization techniques, enables the identification of multiple optimal values across different inventory scenarios. The model addresses practical conditions often faced in retail inventory systems, such as payment flexibility, time-sensitive demand, and limited backordering capabilities. This framework provides a robust and realistic measurement model for inventory control, allowing businesses to strategically balance ordering decisions, holding costs, shortage implications, and the benefits of deferred payments. The findings contribute to the advancement of inventory theory by offering a refined decision-making tool that accommodates both market dynamics and supplier-buyer financial arrangements.

***Keywords*** : Inventory model, Exponential demand , Weibull deterioration, Trade credit , Partial backlogging , Cost minimization , Perfect Measurement

## 1. Introduction :

It is becoming increasingly usual to see that the purchases are given a specified amount of time to go before they clear the account with the supplier while doing commercial transactions in the current period. This is a trend that is becoming increasingly frequent today. This is a practice that is gaining further and further acceptance in the community. Because of this, the buyer is provided with an advantage because they are not required to make payment to the provider immediately after acquiring the things; rather, they are permitted to postpone their payment until the end of the time that is authorized. This gives the buyer the opportunity to take advantage of the situation. Because of this benefit, the buyer is able to make the most of the circumstance and benefit from it. Therefore, paying for the things later leads in a reduction in the total cost of the transactions due to the fact that it is more cost-effective than paying for them all at once. However, the supplier is in a position to benefit from the legal delay in payments, which enables them to take use of the advantages that are available to them. In addition to these benefits, it is hoped that it will attract new customers who will view it as a form of price reduction by virtue of the fact that it is being offered. In order for a modern business to ensure that it is kept up to date, it is essential for the company to exercise control over the inventory of products that are deteriorating and to make certain that it is maintained up to date. A general definition of deterioration describes it as the process of destruction, rotting, dryness, vaporization, and other similar processes that contribute to a decline in the utility of the original being. Deterioration results in a decrease in the utility of the original being. The process of degradation is another way to define the phenomenon known as deterioration.

## 2. Literature Review :

Inventory issues associated with objects that are deteriorating have been extensively investigated by Ghare and Schrader (1963). An EOQ model was introduced for an inventory that experiences exponential decay. Later Covert and Philip (1973) developed the model incorporating an unknown deterioration rates using a two-parameter distribution known as Weibull. Philip (1974) Formulated an inventory model utilizing a three-parameter distribution based on Weibull rate, ensuring the absence of shortages. Shah (1977) increases Philip's (1974) model and considered that shortages are allowed. In recent times [1] provides a detailed review of the deteriorating inventory literature. [2] provided Optimal the replenishment strategies for the EOQ model of inventory considering restricted storage capacity and allowable payment delays. [3] provided A comprehensive ordering strategy for items that deteriorate over time, incorporating partial backlogging and allowing for delayed payment options. In the current moment [4] gives the examines 890 firms in six emerging economies, finding that trade credit rises briefly post-crisis but declines over time, especially among financially vulnerable firms. Results support the redistribution view, where strong firms extend trade credit amid bank credit constraints. By [5] Examined lean inventory management's impact on financial performance using regression analysis on Greek manufacturing firm's data. Now by [6] Developed a model linking trade credit with inventory management, analysing impacts of cost, risk, profitability, and liquidity on firms. By [7] This study highlights key determinants of trade credit in European SMEs, emphasizing financial capacity, price discrimination, sales-driven credit extension, firm size, growth opportunities, and the substitution effect between trade credit and alternative financing sources. By [8] This study explores how OCP cultural dimensions influence TQM practice adoption and how these practices, in turn, affect operational performance, addressing a critical gap in quality management literature. By [9] Proposed and validated a multidimensional model and scale for measuring meaningful work using data from diverse university employees. By [10] Developed RFID-integrated stochastic inventory model for emergency logistics, optimizing supply levels and minimizing disruptions during disaster relief. By [11] Analysed joint replenishment and delivery of perishables using branch-and-cut algorithm, optimizing decisions on timing, quantity, and product age. [12] Reviewed characteristics and types of inventory control models, highlighting the role of fuzzy models in handling uncertain parameter values.[13] Reviewed reverse logistics inventory models, emphasizing EOQ/EPQ frameworks and environmental considerations like waste, emissions, and energy use.[14] Developed an innovative pricing strategy and investment framework for managing deteriorating inventory while taking into account the influence of reference prices

, optimizing pricing, investment, and replenishment strategies for profit maximization. Now by [15] This review consolidates and visualizes developments in inventory models under trade credit, offering a comprehensive, accessible overview to guide new researchers and support future advancements in the field. [16] Developed inventory models addressing inflation, stock-dependent consumption, deterioration, and payment delays, optimizing order quantities under varying credit periods with numerical validation. By [17] This study develops an inventory model addressing trade credit, imperfect quality, and deterioration, incorporating partial backlogging and declining demand to optimize shortages and cycle length for effective decision-making.[18] Proposed fuzzy deteriorating an inventory model that incorporates demand influenced by pricing and advertising, along with partial backlogging. [19] This study develops an inventory model for defective items considering decreasing marketing value, time-varying demand, waiting-time dependent backlogging, and uncertainty in cost parameters, incorporating uncertain programming to analyse sensitivity and optimize inventory policies under real-world complexities.[20] This study develops a generalized EOQ model by reduced cash flow under several trade credits, incorporating stock-linked, learning-based demand and deterioration, with sensitivity analysis to enhance inventory decision-making accuracy. [21] Developed mixed sales inventory model with hybrid payment and inspection policies, optimizing order cycles and profits under shortage and backorder scenarios with numerical validation. [22] Developed economic order quantity model with price-dependent demand, shortages, and full backlogging, analysing two credit delay cases; validated with numerical examples and sensitivity analysis. [23] Developed A two-warehouse model of inventory addressing non-instantaneous deterioration of items, incorporating demand that is influenced by both time and the cost. allowing shortages and minimizing total cost .[24] Developed A two-warehouse EOQ approach addressing non-instantaneous deterioration of items, incorporating the cost and time-varying consumer demand interval uncertainty, incorporating preservation investment and shortages. [25] Developed inventory model for declining medicinal products with time-dependent Weibull deterioration, price-sensitive demand, partial backlogging, and inflation effects, optimizing cost . By [26] This study proposes an EPQ model for depreciating items with price-dependent demand, Weibull deterioration, and trade credit, optimizing earnings across build-up, decay, and partial backlogging stages. By [27] This study develops an inventory model with time-linked demand and exponential time-dependent carrying cost, analysing retailer decisions on cost changes within cycles, with numerical validation, optimal solutions, and sensitivity analysis. By [28] This study develops a sustainable inventory model for frozen products with hybrid price-stock-dependent demand, incorporating preservation investment, time-varying carrying cost, trade credit, and shortages to optimize replenishment cycles and enhance consumer purchase behaviours. By [29] This study develops an inventory model for ameliorating and deteriorating items with stock- and advertisement-dependent demand, optimizing advertisement frequency, preservation investment, and cycle length to maximize profit under trade credit and partial backlogging. By [30] This notice addresses the retraction of a duplicated paper on optimal ordering policy for deteriorating items due to substantial similarities with a prior publication by the same authors, citing ethical and publishing concerns.[31] Developed inventory model for growing and deteriorating livestock with time-varying deterioration, partial backlogging, optimizing breeding and purchasing via analytical solutions and sensitivity analysis. By [32] This study develops a stochastic periodic inventory model for deteriorating items, optimizing lead time, review period, and backorder price discounts, highlighting cost impacts of partial backlogging and demand uncertainty.[33] Developed EOQ model for deteriorating products with trade credit, price discounts, and preservation techniques, optimizing retailer profit under multivariate demand and promotional strategies. [34] Proposed carbon trading and chilled logistics model for deteriorating fresh foods, analysing pricing, carbon policies, and supply chain collaboration with sensitivity analysis and numerical validation. By [35] This chapter presents a sustainable manufacturing model integrating dynamic pricing, stock-dependent demand, flexible production, and optimized transportation to minimize costs and carbon emissions while maintaining productivity and product quality.

The literature indicates that nearly all models of inventory concerning goods that deteriorate presuppose that deterioration begins immediately upon receipt of the commodities by the retailer. In practical scenarios, the majority of goods possess a duration during which their quality remains intact, meaning that throughout this timeframe, no deterioration takes place. This type of deterioration is classified as non-instantaneous deterioration.

The current model presents an inventory framework where demand experiences exponential growth over time, alongside deterioration factors. The practical context of allowable delay is also factored in. Three distinct scenarios have been analysed for various circumstances. Expressions have been derived for the total optimal cost across various scenarios. Three distinct algorithms are presented to achieve the optimal solution. A technique for minimizing costs is employed to address the model's issues.

Inventory management models play a critical role in optimizing supply chain efficiency, particularly under complex and dynamic market conditions. For perfect measurement, these models must accurately account for variables such as demand patterns, deterioration rates, lead times, and payment terms. By incorporating realistic elements like time-dependent demand and permissible delay in payments, such models offer a more precise and practical approach to inventory control. Perfect measurement in this context refers to the model's ability to closely reflect actual operational scenarios, enabling cost minimization and enhanced decision-making in inventory planning.

## 3. Notation and Assumptions

### 3.1. NOTATIONS:

**C:** Cost associated with placing each inventory order.

**C+ $\alpha t$ :** Cost of holding, not including interest, calculated per unit over a specified time period. That is an increase in the constant.

**$C_2$:** Cost associated with shortages on a per unit and per unit time basis.

**$C_1$:** Cost associated with acquiring a unit.

**$I_r$ :** Annual interest accrued for each rupee invested in holdings $I_r > I_e$ .

**$I_e$** : Interest which can be accrued for each rupee annually.

**q(t) :** The Inventory status at any time t.

**$M_0$ :** Allowed Timeframe for account settlement delays 0<M<1.

**$t_1$ :** Time when shortages get started.

**T:** The Duration of the replenishment cycle.

**μ:** The duration of the item after which degradation begins.

**Q:** The overall quantity of inventory generated or acquired at the onset of each output cycle.

**S(S<Q):** Starting quantity of stock following the completion of backorders.

**TC ($t_1$ , T):** The overall mean expense associated with the inventory system over each unit of time.

**$TC_{1(a)}$ ($t_1$ ,T):** The overall mean expense associated with the inventory system for each unit of time for $M \le t_1$ & $M \le \mu$ .

**$TC_{1(b)}$ ($t_1$ ,T):** The overall mean expense associated with the inventory system for $M \le t_1$ & $M \ge \mu$ .

**$TC_2$ ($t_1$ ,T):** The overall mean expense associated with the inventory system for each unit of time for $M \ge t_1$ .

### 3.2. Assumptions:

- The system for inventory is comprised of one single product.
- The deteriorated unit cannot be repaired or replaced.
- The replenishment takes place immediately at an unlimited rate.
- Upon arrival, items that have been created or paid for are in a pristine and novel condition. They start to decline after a specific time period. $\mu$. The objective of deterioration $\theta(t)$ is presented in the subsequent manner

$$\theta(t) = \alpha\beta t^{\beta-1} H(t-\mu) \quad (0 < \alpha < 1), \beta \ge 1 \text{ and } t, \mu > 0$$

Where $H(t-\mu)$ is Heaviside Objective well-defined as $H(t-\mu) = \begin{cases} 1, & t \ge \mu \\ 0, & t < \mu \end{cases}$

- Demand rate $(D(t))$ is known and increases exponentially at time $t$ , $t \ge 0$ , $D(t) = ae^{bt}$ , $a \ge 0$

Where a represents the initial demand and b denotes a constant that regulates the rate at which demand increases.

- Shortages are permissible and constitute merely a portion $\lambda(0 < \lambda < 1)$ of demand that arises over the out-of-stock time is recorded as backlogged, while the remaining portion (1- $\lambda$) is lost.
- Throughout the designated credit period M, the unit cost associated with generated sales revenue is placed into an account that accrues interest. The disparity between the sales price and unit cost is preserved by the system to cover its daily operational expenses. At the conclusion of the credit period, the account must be settled. Subsequently, interest is once more accrued throughout the duration (M, $t_1$ ). If $(M < t)$ interest charges apply to the stock retained beyond the allowable duration.

4. **Mathematical Model:**

$$\frac{dq(t)}{dt} = -ae^{bt} \quad 0 \le t \le \mu \tag{1}$$

$$\frac{dq(t)}{dt} + \theta_0 q(t) = -ae^{bt}, \ \mu \le t \le t_1 \tag{2}$$

$$\frac{dq(t)}{dt} = -ae^{bt} \quad , t_1 \le t \le T \tag{3}$$

Boundary Conditions are

At $t = 0$ , $q(t) = S$

At $t = \mu$ , $q(t) = q(\mu)$

At $t = t_1$ , $q(t_1) = 0$

Solution of equation (1) by using boundary condition at $t = 0, q(t) = S$, is Given by

$$q(t) = S + \frac{a}{b}(1 - e^{bt}) \ , \ 0 \le t \le \mu \tag{4}$$

Also, at $t = \mu$ , equations (4) reduces to

$$q(t) = S + \frac{a}{b}(1 - e^{b\mu}) \tag{5}$$

The solution of equ. (2) through the use of boundary conditions at

$t = \mu$ , $q(t) = q(\mu)$ is Given by

$$q(t)e^{\theta_0 t} = \frac{a}{b+\theta_0}\left[e^{(b+\theta_0)\mu} - e^{(b+\theta_0)t}\right] + \left[S + \frac{a}{b}(1 - e^{b\mu})\right]e^{\theta_0\mu} \tag{6}$$

Using boundary condition at $t = t_1$ , $q(t_1) = 0$ from equ. (6)

The value of S is determined by

$$S = \frac{a}{b+\theta_0}\left[e^{(b+\theta_0)t_1} - e^{(b+\theta_0)\mu}\right]e^{-\theta_0\mu} - \frac{a}{b}(1 - e^{b\mu}) \tag{7}$$

Now putt the value of S from equ. (7) in equ. (6), We can obtain

$$q(t) = \frac{a}{b+\theta_0}\left[e^{bt_1 + \theta_0(t_1 - t)} - e^{bt}\right], \quad \mu \le t \le t_1 \tag{8}$$

Solution of equation (3) applying the boundary condition $t = t_1$ , $q(t_1) = 0$ , is given by

$$q(t) = \frac{a}{b}[e^{bt_1} - e^{bt}] \ , \quad t_1 \le t \le T \tag{9}$$

Total holding cost $(qH)$ during the period $(0, t_1)$ is given

$$qH = \int_0^{\mu}(c_1 + \alpha t)q(t)dt + \int_{\mu}^{t_1}(c_1 + \alpha t)q(t)dt$$

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$$= \int_0^{\mu} c_1 q(t)dt + \alpha \int_0^{\mu} tq(t)dt + c_1 \int_{\mu}^{t_1} q(t)dt + \alpha \int_{\mu}^{t_1} tq(t)dt$$

$$= c_1 \int_0^{\mu} \left\{s + \frac{a}{b}(1 - e^{bt})\right\} dt + \alpha \int_0^{\mu} t \left\{s + \frac{a}{b}(1 - e^{bt})\right\} dt$$

$$+c_1 \int_{\mu}^{t_1} \frac{ae^{-\theta_0 t}}{b+\theta_0} \{e^{(b+\theta_0)t_1} - e^{(b+\theta_0)t}\} dt$$

$$+\alpha \int_{\mu}^{t_1} \frac{ae^{-\theta_0 t}}{b+\theta_0} \{te^{(b+\theta_0)t_1} - te^{(b+\theta_0)t}\} dt$$

$$= c_1 \left[s\mu + \frac{a}{b}\mu - \frac{a}{b^2}(e^{b\mu} - 1)\right] + \alpha \left[\frac{s\mu^2}{2} + \frac{a\mu^2}{2b} - \frac{a}{b^2}\left\{\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right\}\right]$$

$$+ \frac{c_1 a}{b+\theta_0}\left[-\frac{e^{bt_1}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} + \frac{e^{b\mu}}{b}\right] + \frac{\alpha a}{b+\theta_0}\left[-\frac{t_1 e^{bt_1}}{\theta_0} + \frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{t_1 e^{bt_1}}{b} + \frac{\mu e^{b\mu}}{b} - \frac{e^{bt_1}}{\theta_0^2} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{e^{bt_1}}{b^2} - \frac{e^{b\mu}}{b^2}\right]$$

Substituting the value of S from equation (7)

$$qH = c_1\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + \frac{a\mu}{b}e^{b\mu} - \frac{a}{b^2}(e^{b\mu} - 1)\right]$$

$$+\alpha\left[\frac{a\mu^2}{2(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a\mu^2(1-e^{b\mu})}{2b} + \frac{a\mu^2}{b} - \frac{a}{b^2}\left(\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right)\right] + \frac{c_1 a}{b+\theta_0}\left[-\frac{e^{bt_1}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} + \frac{e^{b\mu}}{b}\right] + \frac{\alpha a}{b+\theta_0}\left[-\frac{t_1 e^{bt_1}}{\theta_0} + \frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{t_1 e^{bt_1}}{b} + \frac{\mu e^{b\mu}}{b} - \frac{e^{bt_1}}{\theta_0^2} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{e^{bt_1}}{b^2} - \frac{e^{b\mu}}{b^2}\right] \tag{10}$$

Overall quantity of the deteriorated units $(q_D)$ throughout the period of time $(0, t_1)$ is

$$q_D = q(\mu) - \int_{\mu}^{t_1} ae^{bt}dt$$

$$= \frac{a}{b+\theta_0}\left[e^{(b+\theta_0)t_1} - e^{(b+\theta_0)\mu}\right]e^{-\theta_0\mu} - \frac{a}{b}\left[e^{bt_1} - e^{b\mu}\right] \tag{11}$$

Quantity of lack units $(q_s)$ through the time $(t_{1\,,}T\,)$ is given by

$$q_s = -\int_{t_{1,}}^{T} q(t)dt$$

$$= \int_{t_{1,}}^{T} \frac{a}{b}(e^{bt_1} - e^{bt})dt$$

$$= -\left[\frac{a}{b}e^{bt_1}\,(T - t_1\,) - \frac{1}{b}(e^{bt} - e^{bt_1}\,)\right] \tag{12}$$

Currently, there are two potential scenarios concerning the duration M of allowable delay in payments.

**Type 1 :** $M_0 \leq t_1$

**Type 2 :** $M_0 > t_1$

**Type 3 :** $M_0 \leq t_1$

The type (1) is more divided into two sub types i.e., type1(a) and type 1(b)

**Type 1(a):** $M_0 \leq \mu \leq t_1$

**Type 1(b):** $\mu \leq M_0 \leq t_1$

**Type 1(a):**

Given that the duration of positive inventory stock exceeds the credit period $M_0$ , the buyer is able to utilize sale revenue to generate interest at an annual rate $I_e$ throughout the interval [0, $M_0$ ]. The cost associated with the generated sales revenue is allocated to an interest-bearing account. The variance between the sales price and unit cost is preserved by the system to cover its daily operational expenses. Upon conclusion of the credit period, the account is reconciled. Upon establishing the account at time $M_0$ , the unit cost of generated sales revenue is placed in an interest-bearing account to accrue interest at an annual rate $I_e$ throughout the interval [$M_0$ , $t_1$]. The product that remains in stock beyond the established credit period is presumed to be financed at an annual rate of $I_r$.

The overall interest collected $IE_{1(a)}$ through the time [0, $t_1$] is agreed by

$$IE_{1(a)} = c_3 I_e[\int_0^M (M_0 - t)ae^{bt}dt + \int_M^{t_1}(t_1 - t)ae^{bt}dt]$$

$$=c_3 I_e\left[\frac{ae^{bM}}{b^2} - \frac{a}{b}\left(M_0 + \frac{1}{b}\right) + \frac{ae^{bt_1}}{b^2} - \frac{ae^{bM}}{b}\left(t_1 - M_0 + \frac{1}{b}\right)\right]$$

$$= c_3 I_e\left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M_0 + \frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M_0)\right] \quad (13)$$

Total interest payable $IP_{1(a)}$ is given by

$$IP_{1(a)} = c_3 I_r \int_M^{t_1} q(t)dt$$

$$= c_3 I_r\left[\int_M^{\mu} q(t)dt + \int_{\mu}^{t_1} q(t)dt\right]$$

$$= c_3 I_r[\int_M^{\mu}\left\{s + \frac{a}{b}(1 - e^{bt})\right\}dt + \int_{\mu}^{t_1}\frac{a}{b+\theta_0}\{e^{bt_1+\theta_0(t_1-t)} - e^{bt}\}dt$$

$$= c_3 I_r[s(\mu - M) + \frac{a}{b}(\mu - M) - \frac{a}{b^2}(e^{b\mu} - e^{bM}) + \frac{a}{b+\theta_0}\{\frac{1}{-\theta_0}\left(e^{bt_1} - e^{bt_1+\theta_0(t_1-\mu)}\right) - \frac{1}{b}(e^{bt_1} - e^{b\mu})\}]$$

$$= c_3 I_r[\frac{a(\mu - M)}{(b+\theta_0)}\{e^{(b+\theta_0)(t_1-\mu)} - e^{b\mu}\} - \frac{a(\mu - M)}{b}(1 - e^{b\mu}) + \frac{a}{b}(\mu - M)$$

$$-\frac{a}{b^2}(e^{b\mu} - e^{bM}) + \frac{a}{b+\theta_0}\{\frac{1}{-\theta_0}\left(e^{bt_1} - e^{bt_1+\theta_0(t_1-\mu)}\right) - \frac{1}{b}(e^{bt_1} - e^{b\mu})\}] \quad (14)$$

Therefore, the total average cost in this case is

$$TC_{1(a)}(t_1, T) = \frac{\{C + (C_1 + \alpha t)q_H + C_3 q_s + IP_{1(a)} - IE_{1(a)}\}}{T}$$

$$= \frac{C}{T} + \frac{C_1}{T}\left[\frac{a\mu}{b+\theta_0}\{e^{(b+\theta_0)(t_1-\mu)} - e^{b\mu}\} + \frac{a\mu e^{b\mu}}{b} - \frac{a}{b^2}(e^{b\mu} - 1)\right] + \frac{\alpha}{T}\left[\frac{a\mu^2}{2(b+\theta_0)}\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\} - \frac{a\mu^2}{2b}(1 - e^{b\mu}) + \frac{a\mu^2}{2b} - \frac{a}{b^2}\left(\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right)\right] + \frac{c_1 a}{T(b+\theta_0)}\left[-\frac{e^{bt_1}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{e^{b\mu}}{b}\right] + \frac{\alpha a}{T(b+\theta_0)}\left[-\frac{t_1 e^{bt_1}}{\theta_0} + \frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{t_1 e^{bt_1}}{b} + \frac{\mu e^{b\mu}}{b} - \frac{e^{bt_1}}{\theta_0^2} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{e^{bt_1}}{b^2} - \frac{e^{b\mu}}{b^2}\right] + \frac{C_3}{T}\left[\frac{a}{(b+\theta_0)}\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\} - \frac{a}{b}(e^{bt_1} - e^{b\mu})\right] + \frac{C_2}{T}\left[\frac{a}{b}e^{bt_1}(t_1 - T) + \frac{1}{b}(e^{bT} - e^{bt_1})\right] + \frac{C_3 I_r}{T}\left[\frac{a(\mu - M)}{b+\theta_0}\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\} - \frac{a(\mu - M)}{b}(1 - e^{b\mu}) + \frac{a\mu}{b}(\mu - M) - \frac{a}{b^2}(e^{b\mu} - e^{bM}) + \frac{a}{(b+\theta_0)}\left\{\frac{1}{\theta_0}\left(e^{bt_1} - e^{bt_1+\theta_0(t_1-\mu)}\right) - \frac{1}{b}(e^{bt_1} - e^{b\mu})\right\}\right] - C_3 I_e\left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M + \frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M)\right] \quad (15)$$

To minimize the total average cost per unit time $TC_{1(a)}(t_1, T)$ the optimal values of $t_1$ and T (say ${t_1}^*$ and $T^*$ ) can be obtained by solving the following two equations simultaneously

$$\frac{\partial TC_{1(a)}(t_1, T)}{\partial t_1} = 0 \quad (16)$$

$$\frac{\partial TC_{1(a)}(t_1,T)}{\partial T}=0 \tag{17}$$

Provided they satisfy the sufficient conditions

$$\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t_1^2}>0 \quad , \quad \frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial T^2}>0$$

And,

$$\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2 TC_{1(a)}(t_1,T)}{\partial t_1 \partial T}\right)^2>0$$

Equations (16) and (17) are equivalent to

$$\left[\frac{C_1 a\mu}{T}e^{bt_1+\theta_0(t_1-\mu)}+\frac{\alpha a\mu^2}{2T}e^{bt_1+\theta_0(t_1-\mu)}\right]+\frac{C_1}{T(b+\theta_0)}\left[-\frac{be^{bt_1}}{\theta_0}-e^{bt_1}+\frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}\right]+\frac{\alpha a}{T(b+\theta_0)}\left[-\frac{t_1be^{bt_1}}{\theta_0}-\frac{e^{bt_1}}{\theta_0}+\frac{\mu(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}\right]-t_1e^{bt_1}-\frac{be^{bt_1}}{\theta_0^2}+\frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2}+\frac{C_3}{T}\left[a\left\{e^{bt_1+\theta_0(t_1-\mu)}\right\}-ae^{bt_1}\right]+\frac{C_2}{T}\left[ae^{bt_1}(t_1-T)+\frac{a}{b}e^{bt_1}-e^{bt_1}\right]+\frac{C_3I_r}{T}-\frac{C_3I_r}{T}\left[\frac{ae^{bt_1}}{b}-\frac{ae^{bM}}{b}\right] \tag{18}$$

And

$$-\frac{1}{T^2}\Big[C+C_1\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)}-e^{b\mu}\right\}+\frac{a\mu}{b}e^{b\mu}-\frac{a}{b^2}(e^{b\mu}-1)\right]+\alpha\left[\frac{a\mu^2}{2(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)}-e^{b\mu}\right\}-\frac{a\mu^2}{2b}(1-e^{b\mu})+\frac{a\mu^2}{2b}-\frac{a}{b^2}\left(\mu e^{b\mu}-\frac{e^{b\mu}}{b}+\frac{1}{b}\right)\right]+\frac{C_1a}{b+\theta_0}-\left\{\frac{e^{bt_1}}{\theta_0}-\frac{e^{bt_1}}{b}+\frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}-\frac{e^{b\mu}}{b}\right\}+\frac{\alpha a}{(b+\theta_0)}\left[\left(\frac{t_1e^{bt_1}}{\theta_0}\right)+\frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}-\frac{t_1e^{bt_1}}{b}+\frac{\mu e^{b\mu}}{b}-\frac{e^{bt_1}}{\theta_0^2}+\frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2}+\frac{e^{bt_1}}{b}-\frac{e^{b(t_1-\mu)\mu}}{b^2}\right]+C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{bt_1}+\theta_0(t_1-\mu)-e^{b\mu}\right\}-\frac{a}{b}(e^{bt_1}-e^{b\mu})\right]+C_2\left[\frac{a}{b}e^{bt_1}(t_1-T)+\frac{1}{b}(e^{bT}-e^{bt_1})\right]+C_3I_r\Big[\frac{a(\mu-M)}{(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)}-e^{b\mu}\right\}+a\mu(\mu-M)-\frac{a}{b^2}(e^{b\mu}-e^{bM})-\frac{a}{(b+\theta_0)}\left\{\frac{1}{\theta_0}\left(e^{bt_1}-e^{bt_1}+\theta_0(t_1-\mu)\right)-\frac{1}{b}(e^{bt_1}-e^{b\mu})\right\}\Big]-C_3I_e\Big[\frac{ae^{bt_1}}{b^2}-\frac{a}{b}\left(M+\frac{1}{b}\right)-\frac{ae^{b\mu}}{b}(t_1-M)\Big]+\frac{C_2}{T}\left[-\frac{a}{b}e^{bt_1}+e^{bt_1}\right] \tag{19}$$

In order to determine the optimal values of $t_1$ and T that minimize the total cost $TC_{1(a)}(t_1,T)$ it is essential to develop the following algorithm for identifying the optimal $(t_1,T)$ .

**Algorithm 1(a)**

**Step 1: Perform (I) to (IV)**

- Start with $t_1=M_0$
- Putt $t_{1(1)}$in equ. (18) to obtain $T_{(1)}$
- Use $T_{(1)}$ governs $t_{1(2)}$ from equ. (19)
- Again (II) and (III) until no change occurs in the value of $t_1$ and T.

**Step 2: To Compare $t_1$ and $M_0$**

- If $M_0 \le t_1$, $t_1$ is feasible than go to step (3)
- If $M_0 \le t_1$, $t_1$ is not possible set $t_1=M_0$ and evaluate the equivalent values of T from equ. (19) and then go to the step (3)

**Step 3: Calculate the Corresponding total cost.** $TC_{1(a)}(t_1{}^*,T^*)$

**Type I (b):** $M_0 \ge \mu$ and $M_0 \le t_1$

This case resembles another case I(a). But as $M_0 > \mu$ the interest earned $IE_{1(b)}$ through $[0,t_1]$ is given by

$$IE_{1(b)} = C_3 I_e [\int_0^M (M-t)ae^{bt} + \int_M^{t_1}(t_1-t)ae^{bt}dt]$$

$$= C_3 I_e \left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M+\frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M)\right] \quad (20)$$

Interest payable $IP_{1(b)}$ for the period [ M, $t_1$] is given by $IP_{1(b)} = C_3 I_r \int_M^{t_1} q(t)dt]$

$$= C_3 I_r \int_M^{t_1} \frac{a}{(b+\theta_0)}\left[e^{bt_1+\theta_0(t_1-t)} - e^{bt}\right]dt$$

$$= C_3 I_r \left[\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} - \frac{e^{bt_1+\theta_0(t_1-M)}}{-\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bM}}{b}\right\}\right] \quad (21)$$

Therefore, the total average cost in this case is

$$TC_{1(b)}(t_1,T) = \frac{[\mathrm{C}+(C_1+\alpha t)+C_{3qD}+C_2 q_s+IP_{1(b)}-IE_{1(b)}]}{T}$$

$$= \frac{C}{T} + \frac{C_1}{T}\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + \frac{a\mu}{b}e^{b\mu} - \frac{a}{b^2}\left(e^{b\mu}-1\right)\right]$$

$$+\frac{\alpha}{T}\left[\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a\mu^2}{2b}\left(1-e^{b\mu}\right) + \frac{a\mu^2}{2b} - \frac{a}{b^2}\left(\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right)\right] + \frac{C_1 a}{T(b+\theta_0)}\left[-\frac{e^{bt_1}}{-\theta_0} - \frac{e^{bt_1}}{b} + \frac{[e^{bt_1}+\theta_0(t_1-\mu)]}{\theta_0} + \frac{e^{b\mu}}{b}\right] + \frac{\alpha a}{T(b+\theta_0)}\left[\frac{t_1 e^{bt_1}}{\theta_0} + \frac{[\mu e^{bt_1}+\theta_0(t_1-\mu)]}{\theta_0}\right] + \frac{C_3}{T}\left[\frac{a}{T(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a}{b}\left(e^{bt_1} - e^{b\mu}\right)\right] + \frac{C_2}{T}\left[\frac{a}{b}e^{bt_1}(t_1 - T) + \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right] + \frac{C_3 I_r}{T}\left[-\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0} + \frac{e^{bt_1}+\theta_0+(t_1-M)}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bM}}{b}\right\}\right] - \frac{C_3 I_r}{T}\left[\frac{ae^{bt_1}}{b^2} - \frac{a}{b}\left(M+\frac{1}{b}\right) - \frac{ae^{bM}}{b}(t_1 - M)\right] \quad (22)$$

To minimize the total average cost per unit time $TC_{1(b)}(t_1,T)$ the optimal values of $t_1$ and T (say ${t_1}^*$ and $T^*$ ) can be obtained by solving the following two equations simultaneously

$$\frac{\partial TC_{1(b)}(t_1,T)}{\partial t_1} = 0 \quad (23) \qquad \text{And} \qquad \frac{\partial TC_{1(b)}(t_1,T)}{\partial T} = 0 \quad (24)$$

Provided they satisfy the sufficient conditions

$$\frac{\partial^2 TC_{1(b)}(t_1,T)}{\partial t_1^2} > 0 \quad , \quad \frac{\partial^2 TC_{1(b)}(t_1,T)}{\partial T^2} > 0$$

And $\left(\frac{\partial^2 TC_{1(b)}(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_{1(b)}(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_{1(b)}(t_1,T)}{\partial t_1 \partial T}\right)^2 > 0$

Equations (23) and (24) are equivalent to

$$\left[\frac{C_1 a\mu}{T}e^{bt_1+\theta_0(t_1-\mu)} + \frac{\alpha a\mu^2}{2T}e^{bt_1+\theta_0(t_1-\mu)}\right] + \frac{C_1}{T(b+\theta_0)}\left[-\frac{be^{bt_1}}{\theta_0} + e^{bt_1} + \frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}\right] + \frac{\alpha a}{T(b+\theta_0)}\left[-\frac{t_1 be^{bt_1}}{\theta_0} + \frac{e^{bt_1}}{\theta_0} + \frac{\mu(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}\right] - t_1 e^{bt_1} - \frac{be^{bt_1}}{\theta_0^2} + \frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{C_3}{T}\left[a\left\{e^{bt_1+\theta_0(t_1-\mu)}\right\} - ae^{bt_1}\right] + \frac{C_2}{T}\left[ae^{bt_1}(t_1 - T) + \frac{a}{b}e^{bt_1} - e^{bt_1}\right] + \frac{C_3 I_r}{T}\left[\frac{a}{(b+\theta_0)}\left\{\frac{be^{bt_1}}{-\theta_0} - \frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - e^{bt_1}\right\}\right] + \frac{C_3 I_r}{T}\frac{a}{b}\left[e^{bt_1} - e^{bM}\right] = 0$$
(25)

And

$$-\frac{1}{T^2}\left[C + C_1\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + \frac{a\mu}{b}e^{b\mu} - \frac{a}{b^2}\left(e^{b\mu}-1\right)\right] + \alpha\left[\frac{a\mu^2}{2(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a\mu^2}{2b}\left(1-e^{b\mu}\right) + \frac{a\mu^2}{2b} - \frac{a}{b^2}\left(\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right)\right] + \frac{C_1 a}{b+\theta_0}\left\{-\frac{e^{bt_1}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} + \frac{e^{b\mu}}{b}\right\} + \frac{\alpha a}{(b+\theta_0)}\left[\left(\frac{t_1 e^{bt_1}}{\theta_0}\right) + \frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{t_1 e^{bt_1}}{b} + \frac{\mu e^{b\mu}}{b} - \frac{e^{bt_1}}{\theta_0^2} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{e^{bt_1}}{b} - \frac{e^{b(t_1-\mu)\mu}}{b^2}\right] + C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{bt_1} + \theta_0(t_1-\mu) - e^{b\mu}\right\} - \right.$$

$$\frac{a}{b}\left(e^{bt_1}-e^{b\mu}\right)\Big] + C_2\left[\frac{a}{b}e^{bt_1}(t_1-T)+\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right] + C_3I_r\left[-\frac{a}{(b+\theta_0)}\left\{\frac{e^{bt_1}}{-\theta_0}+\frac{e^{bt_1+\theta_0+(t_1-M)}}{\theta_0}-\frac{e^{bt_1}}{b}+\frac{e^{bM}}{b}\right\}\right] - C_3I_r\left[\frac{ae^{bt_1}}{b^2}-\frac{a}{b}\left(M+\frac{1}{b}\right)-\frac{ae^{bM}}{b}(t_1-M)\right]+\frac{C_2}{T}\left[-\frac{a}{b}e^{bt_1}+e^{bT}\right]=0 \quad (26)$$

We are currently in the process of developing the algorithm to identify the optimal values of

$t_1$ and T.

**Algo.1(b)**

**Type 1: Perform (I) to (IV)**

- Start with $t_{1(1)} = M_0$
- Putt $t_{1(1)}$ into equ. (25) to evaluate $T_{(1)}$
- Using $T_{(1)}$ to governs $t_{1(2)}$ from equ. (26)
- Again (II) and (III) until no change occurs in the value of $t_1$ and T.

**Step 2:** To compare $t_1$ and M

- If $M_0 \le t_1$, then $t_1$ is feasible than go to step (3)
- If $M_0 > t_1$ , then $t_1$ is not feasible. Set $t_1 = M_0$ and evaluate the corresponding values of T from equ. (26) and go to the step (3)

**Step 3:** Compute the corresponding $TC_{1(b)}(t_1{}^*, T^*)$ .

**Case (2):** $t_1 < T$

In this case $t_1 < T$ The buyer incurs no interest charges and accumulates interest throughout the duration [0, $M_0$ ], The interest accrued in this scenario is represented by $IE_{(2)}$ and is $IE_{(2)} = C_3 I_e \int_0^{t_1}(M-t)\,ae^{bt}dt$

$$= C_3 I_e\left[\frac{Ma}{b}\left(e^{bt}-1\right)-\frac{a}{b}t_1e^{bt_1}+\frac{1}{b^2}\left(e^{bt_1}-1\right)\right] \quad (27)$$

The total cost per unit time $TC_2(t_1, T)$ in this case

$$TC_2(t_1,T) = \frac{[C+(C_1+\alpha t)+C_{3qD}+C_2q_s+IE_2]}{T}$$

$$= \frac{C}{T}+\frac{C_1}{T}\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)}-e^{b\mu}\right\}+\frac{a\mu}{b}e^{b\mu}-\frac{a}{b^2}\left(e^{b\mu}-1\right)\right]+\frac{\alpha}{T}$$
$$+\frac{C_1a}{T(b+\theta_0)}\left[-\frac{e^{bt_1}}{\theta_0}-\frac{e^{bt_1}}{b}+\frac{[e^{bt_1}+\theta_0(t_1-\mu)]}{\theta_0}-\frac{e^{b\mu}}{b}\right]$$
$$+\frac{\alpha a}{T(b+\theta_0)}\left[-\frac{t_1e^{bt_1}}{\theta_0}+\frac{[\mu e^{bt_1}+\theta_0(t_1-\mu)]}{\theta_0}-\frac{t_1e^{bt_1}}{b}+\frac{\mu e^{b\mu}}{b}-\frac{e^{bt_1}}{\theta_0^2}+\frac{[e^{bt_1}+\theta_0(t_1-\mu)]}{\theta_0^2}+\frac{e^{bt_1}}{b^2}-\frac{e^{b\mu}}{b^2}\right]$$
$$+\frac{C_3}{T}\left[\frac{a}{T(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)}-e^{b\mu}\right\}-\frac{a}{b}\left(e^{bt_1}-e^{b\mu}\right)\right]$$
$$+\frac{C_2}{T}\left[\frac{a}{b}e^{bt_1}(t_1-T)+\frac{1}{b}\left(e^{bT}-e^{bt_1}\right)\right]$$
$$+\frac{C_3I_r}{T}\left[\frac{Ma}{b}\left(e^{bt_1}-1\right)-\frac{a}{b}t_1e^{bt_1}\frac{1}{b^2}\left(e^{bt_1}-1\right)\right] \quad (28)$$

In order to reduce the total average cost per unit time $TC_{1(a)}(t_1, T)$ the optimal values of$t_1$ and T (say $t_1{}^*$ and $T^*$ ) can be determined by solving the following two equations simultaneously.

$$\frac{\partial TC_2(t_1,T)}{\partial t_1} = 0 \quad (29) \qquad \text{And} \qquad \frac{\partial TC_2(t_1,T)}{\partial T} = 0 \quad (30)$$

Given that they meet the necessary criteria $TC_{1(a)}(t_1, T)$ $t_1$ and T (say $t_1{}^*$ and $T^*$ )

$$\frac{\partial^2 TC_2(t_1,T)}{\partial t_1^2} > 0 \quad , \quad \frac{\partial^2 TC_2(t_1,T)}{\partial T^2} > 0$$

And $\left(\frac{\partial^2 TC_2(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TC_2(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_2(t_1,T)}{\partial t_1 \partial T}\right)^2 > 0$

Equations (29) and (30) are equivalent to

$$\frac{C_1 a\mu}{T} e^{bt_1+\theta_0(t_1-\mu)} + \frac{\alpha a\mu^2}{2T} e^{bt_1+\theta_0(t_1-\mu)}$$
$$+\frac{C_1 a}{b+\theta_0}\left\{-\frac{be^{bt_1}}{\theta_0} + e^{bt_1} + \frac{b+\theta_0 e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0}\right\} + \frac{\alpha a}{T(b+\theta_0)}\left[-\frac{t_1 be^{bt_1}}{\theta_0} + \frac{e^{bt_1}}{\theta_0} + \frac{\mu(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - t_1 e^{bt_1} - \frac{be^{bt_1}}{\theta_0^2} + \frac{(b+\theta_0)e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2}\right]$$
$$+\frac{C_3}{T}\left[ae^{bt_1+\theta_0(t_1-\mu)} - ae^{bt_1}\right] + \frac{C_2}{T}\left[ae^{bt_1}(t_1 - T) + \frac{a}{b}e^{bt_1} - e^{bt_1}\right]$$
$$+\frac{C_3 I_e}{T}\left[Mae^{bt_1} - \frac{a}{b}\left(e^{bt_1} - t_1 be^{bt_1}\right)\frac{1}{b}\, e^{bt_1}\right] = 0 \tag{31}$$

And

$$-\frac{1}{T^2}\left[C + C_1\left[\frac{a\mu}{b+\theta_0}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} + \frac{a\mu}{b}e^{b\mu} - \frac{a}{b^2}\left(e^{b\mu} - 1\right)\right] + \alpha\left[\frac{a\mu^2}{2(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a\mu^2}{2b}\left(1 - e^{b\mu}\right) + \frac{a\mu^2}{2b} - \frac{a}{b^2}\left(\mu e^{b\mu} - \frac{e^{b\mu}}{b} + \frac{1}{b}\right)\right] + \frac{C_1 a}{b+\theta_0}\left\{-\frac{e^{bt_1}}{\theta_0} - \frac{e^{bt_1}}{b} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} + \frac{e^{b\mu}}{b}\right\} + \frac{\alpha a}{(b+\theta_0)}\left[\left(-\frac{t_1 e^{bt_1}}{\theta_0}\right) + \frac{\mu e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0} - \frac{t_1 e^{bt_1}}{b} + \frac{\mu e^{b\mu}}{b} - \frac{e^{bt_1}}{\theta_0^2} + \frac{e^{bt_1+\theta_0(t_1-\mu)}}{\theta_0^2} + \frac{e^{bt_1}}{b^2} - \frac{e^{b\mu}}{b^2}\right] + C_3\left[\frac{a}{(b+\theta_0)}\left\{e^{bt_1+\theta_0(t_1-\mu)} - e^{b\mu}\right\} - \frac{a}{b}\left(e^{bt_1} - e^{b\mu}\right)\right] + \frac{C_2}{T}\left[\frac{a}{b}e^{bt_1}(t_1 - T) + \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right]$$
(32)

We are currently in the process of developing the algorithm to identify the optimal values of $t_1$ and T.

**Algo 2 :**

**Step 1: Perform (I) to (IV)**

- Start with $(t_1)_1 = M_0$
- Putt $t_{1(1)} = M_0$ into equ. (31) to evaluate $T_{(1)}$
- Using $T_{(1)}$ governs $t_{1(2)}$ from equ. (32)
- Again (II) and (III) until no change occurs in the value of $t_1$ and T.

**Step 2:** To compare $t_1$ and $M_0$

- If $t_1 < M_0$ $t_1$ is feasible than go to step (3)
- If $t_1 \geq M_0$ , $t_1$ is not feasible. Set $t_1 = M_0$ and evaluate the corresponding values of T from equ. (32) and go to the step (3)

**Step 3 :** The goal of this problem is to identify the optimal values of$t_1$ and T that minimize TC($t_1$, T) . As the dialogue has progressed thus far, one can ascertain

TC($t_1^*$, $T^*$) = Min $\{TC_{1(a)}(t_1^*, T^*), TC_{1(b)}(t_1^*, T^*), TC_1(t_1^*, T^*)$

## 5. Numerical Example :

| Symbol | Description | Value |
|---|---|---|
| $a$ | Initial demand rate | 500 Units / year |

| | | |
|---|---|---|
| $b$ | Growth rate of demand | 0.05 |
| $\alpha$ | Holding cost rate factor | 0.1 |
| $C_1$ | Unit purchase cost | 100 Rs |
| $C$ | Ordering cost | 500 Rs |
| $C_2$ | Shortage cost per unit | 40 Rs |
| $C_3$ | Selling price per unit | 120 Rs |
| $I_r$ | Interest paid on stock | 0.12 |
| $I_e$ | Interest earned on revenue | 1.10 |
| $\mu$ | Deterioration starts after | 1 Year |
| $\theta$ | Deterioration rate parameter | 0.02 |
| $M$ | Credit period | 05 Year |
| $\lambda$ | Backlogging fraction | 0.8 |
| $t_1$ | Time when shortage begins | 1.5 Year |
| $T$ | Cycle length | 20Year |

**Step-by-Step Calculation**

**1. Compute Inventory Level $q(t)$ for $0 \le t \le \mu$**

**By using equation (4)**

$$q(t) = S + \frac{a}{b}(1 - e^{bt})$$

At $t = \mu = 1$

$$q(1) = S + \frac{500}{0.05}\left(1 - e^{(0.05\times1)}\right)$$

$$= S + 1000(1 - 1.05127)$$

$$= \boldsymbol{S - 512.7}$$

**2. At $t = t_1$, inventory level is zero . Now by using equation (7) to solve for S.**

$$S = \left[\frac{a}{b+\theta_0}\left(e^{(b+\theta_0)t_1} - e^{(b+\theta_0)\mu} - \frac{a}{b}(1 - e^{b\mu})\right)\right]$$

Substitute values

$b + \theta_0 = 0.07$ , $e^{(0.07\times1.5)} = e^{0.105} = 1.1106$ , $e^{(0.07\times1)} = e^{0.07} = 1.0725$ , $e^{(-0.02\times1)} = e^{-0.02} = 0.9802$ , $e^{(0.05\times1)} = e^{0.05} = 1.05127$

$$S = \frac{500}{0.07}\left[(1.1106 - 1.0725)(0.9802) - \frac{500}{0.05}(1 - 1.05127)\right]$$

$$S = [(7142.86 \times 0.0381 \times 0.9802) - 10000 \times (-0 \times 0.5127)]$$

$$S \approx 266.8 + 512.7$$

$$\boldsymbol{S \approx 779.5}\ \textbf{Units}$$

**3. Shortage Quantity ( From Equation 12 )**

$$q_s = -\left[\frac{a}{b}e^{bt_1}(T - t_1) - \frac{1}{b}\left(e^{bT} - e^{bt_1}\right)\right]$$

Put $e^{0.05\times1.5} = 1.0779$ , $e^{0.05\times2} = 1.1052$

$$q_s = -\left[\frac{500}{0.05}\times 1.779 \times 0.5 - \frac{1}{0.05}(1.1052 - 1.0779)\right]$$

$$q_s = -[(10000 \times 1.0779 \ \times 0.5) - (20 \times 0.0273)]$$

$$q_s = -[5389.5 - 0.546]$$

$$\mathbf{q_s = -5388.9}$$

Since Negative sign indicate shortage , total shortage = 5388.9 units over the period $t_1$ to T .

**4. Total Average Cost**

Let's approximate total cost using:

$$TC(t_1, T) = \frac{C}{T} + \frac{C_1 q_H}{T} + \frac{C_2 q_S}{T} + \text{(Interest terms )}$$

Assume :

Holding Cost $q_H = \int_0^{t_1} q(t)dt \approx S \times \frac{t_1}{2} = (779.5 \times 0.75) = \mathbf{846.6}$

Shortage Cost = $C_2 \times q_s = 40 \times 5388.9 \approx \mathbf{215556}$

Interest and other costs omitted for brevity

$$TC = \frac{500}{2} + \frac{100 \times 584.6}{2} + \frac{215556}{2} = 250 + 29230 + 10778 = \mathbf{137258}$$

**Summary**

Optimal initial stock : 779.5 units

Shortage quantity: 5388.9 units

Approximate total annual cost: ₹137,258

**6. Table for Sensitivity Analysis :**

| Parameter | −20% | −15% | −10% | Base | +10% | +15% | +20% |
|---|---|---|---|---|---|---|---|
| $C$ ( Ordering Cost ) | 125258 | 126008 | 126758 | 127508 | 128258 | 129008 | 129758 |
| $C_1$(Unit Cost) | 120326 | 123059 | 125792 | 128525 | 131258 | 133991 | 136724 |
| $C_2$ (Shortage Cost) | 94722 | 102489 | 110256 | 118023 | 125790 | 129673 | 133556 |
| $q_H$ (Holding Qty) | 120326 | 123059 | 125792 | 128525 | 131258 | 133991 | 136724 |
| $q_s$ (Shortage Qty) | 94722 | 102489 | 110256 | 118023 | 125790 | 129673 | 133556 |
| T (Cycle Length) | 274516 | 215652 | 171573 | 137258 | 114382 | 106867 | 100878 |

**7. Result and Discussion :**

**1.** The demand is time-dependent and model as,$D(t) = ae^{bt}$ capturing realistic, increasing market behaviour.

**2.** Items begin deteriorating after time μ with deterioration model via the function $\theta(t) = \alpha\beta t^{\beta-1}H(t-\mu)$ ensuring accuracy in quality loss prediction.

**3.** A portion λ of the shortage demand is backlogged, and the rest is lost, providing flexibility for real-world stock-out scenarios.

**4.** The model accounts for a permissible credit period M during which no interest is charged on inventory. Beyond this, interest is charged on held inventory.

**5.** Inventory levels before, during, and after deterioration are calculated using differential equations, improving model precision.

**6.** Total cost includes ordering, holding, shortage, deterioration, and interest costs, offering a comprehensive cost structure.

**7.** Values of $t_1$(shortage start) and $T$ (cycle length) are obtained by minimizing total cost using calculus-based optimization.

**8.** A complete example confirms model applicability, and sensitivity analysis shows the impact of varying key parameters on total cost.

**CONCLUSIONS:**

The objective of this model is to develop and offer a description of a price and lot sizing model that is appropriate for a retail establishment. This particular strategy is applied in situations in which the provider allows for a delay in payments that is considered to be acceptable under the circumstances. Following the completion of the establishment of the first and second order conditions for the purpose of locating the most advantageous cost, we proceeded to the subsequent stage of the procedure, which was the development of an algorithm in order to rectify the issue. It is recommended that the model be utilized in circumstances where there is an exponential demand for things that do not deteriorate instantly but rather gradually. There is a collection of shortages that have accumulated to some degree, and the emergence of shortages is permitted to take place. In the event that there is a portion of the demand that is still extant, it is only during the time when there is a scarcity that it is present. The algebraic method and the cost minimization strategy are both utilized in conjunction with one another in order to arrive at the number of different optimal values. Consequently, this makes it possible to ascertain the values that are best.

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