Product Edge-Antimagic Vertex Labeling of Graphs

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Abstract:- A (p, q)-graph G, is defined to be *product antimagic* if there is a labeling from $E(G)$ onto $\{1, 2, . . . , q\}$ such that, at each vertex u, the product of the labels on the edges incident with u are distinct. Similarly, a (p, q)-graph G is defined to be *product edge-antimagic* if there is a labeling f from $V(G) \bigcup E(G)$ onto $\{1, 2, . . . , p + q\}$ with the property that the value $f(u).f(v).f(uv)$, for any edge $uv \in E(G)$ are distinct. In this paper, we introduce the *product edge-antimagic vertex (PEAV) labeling* as a bijection f from $V(G)$ to $\{1, 2, . . . , p\}$ such that, for any edge $uv \in E(G)$, the product $f(u).f(v)$ are distinct. Also we have proved the existence of PEAV labeling for paths and cycles.

Keywords: *Graph Labeling, Product Magic Labeling, Product Antimagic Labeling.*

1. Introduction

By a *graph G* we mean a finite, undirected, connected graph without any loops or multiple edges. Let V(G) and E(G) be the set of vertices and edges of a graph G respectively. The order and size of a graph G is denoted as $p = |V(G)|$ and $q = |E(G)|$. For general graph theoretic notions we refer Harary [3].

A *labeling* of a graph is an assignment of numbers (usually positive or non-negative integers) to the vertices (a *vertex labeling*) or to the edges (an *edge labeling*) or to the combined set of vertices and edges (a *total labeling*) of the graph. There are many types of labelings and a detailed survey of many of them can be found in the dynamic survey of graph labeling by J.A. Gallian [2].

The *edge weight* of an edge uv, denoted by $\Lambda(uv)$, is defined as the sum of labels of the graph elements associated with uv. That is, if f is an edge labeling, then $\Lambda(uv) = f(uv)$; if f is a vertex labeling, then $\Lambda(uv) = f(u) + f(v)$; and if f is a total labeling, then $\Lambda(uv) = f(u) + f(uv) + f(v)$.

Similarly the *vertex weight* of a vertex v, denoted by $\Lambda(v)$, is defined as the sum of labels of the graph elements associated with v. That is, if f is a vertex labeling, then $\Lambda(\mathrm{v}) = \sum_{u\in N(v)} f(u)$; if f is an edge labeling, then $\Lambda(\mathrm{v}) = \sum_{u\in N(v)} f(uv)$; and if f is a total labeling, then $\Lambda(\mathrm{v}) = f(v) + \sum_{u\in N(v)} f(uv)$.

In 1970 Kotzig and Rosa [6] defined an *magic valuation* of a graph G as a bijection f from $V(G) \bigcup E(G)$ to the set $\{1, 2, . . . , |V(G)| + |E(G)|\}$ such that for each edge $uv \in E(G)$, the edge weight $\Lambda(uv) = f(u) + f(uv) + f(v)$ is a constant. This notion was rediscovered by Ringel and Llodo [8] in 1996 who called this labeling as *edge-magic*.

As a natural extension of the notion of edge magic total labeling, Hartsfield and Ringel [4] introduced the concept of an *antimagic labeling* and they defined an *antimagic labeling* of a (p, q)-graph G as a bijection f from $E(G)$ to the set $\{1, 2, . . . , q\}$ such that the sums of label of the edges incident to each vertex $v \in V(G)$ are distinct.

In 2000, Figueroa-Centeno, Ichishima, and Muntaner-Batle [1] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph G of size q to be *product magic* if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q to be *product antimagic* if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$, such that the products of the labels on the edges incident at each vertex v are distinct. They proved the following: a graph of size q is *product magic* if and only if $q \leq 1$ (that is, if and only if it is K_2 or $\overline{K_n}$ or $K_2 \cup \overline{K_n}$); every path P_n ($n \geq 4$) is product antimagic; every cycle C_n is product antimagic; every 2-regular graph is product antimagic; and, if G is product antimagic, then so are $G + K_1$ and $G \odot \overline{K_n}$. They conjectured that a connected graph of size q is *product antimagic* if and only if $q \geq 3$.

Kaplan et al. [5] proved that the following graphs are *product antimagic*: the disjoint union of cycles and paths where each path has at least three edges; connected graphs with n vertices and m edges where $m \geq 4n\ln n$; all complete k-partite graphs except K_2 and $K_{1,2}$. In [7], Pikhurko characterizes all large graphs that are *product antimagic*.

Figueroa-Centeno, Ichishima and Muntaner-Batle [1] also defined a graph G with p vertices and q edges to be *product edge-magic* if there is a labeling f from $V(G) \bigcup E(G)$ onto $\{1, 2, . . . , p + q\}$ such that the value $f(u).f(v).f(uv)$, for any edge $uv \in E(G)$ is a constant and *product edge-antimagic* if there is a labeling f from $V(G) \bigcup E(G)$ onto $\{1, 2, . . . , p + q\}$ with the property that the value $f(u).f(v).f(uv)$, for any edge $uv \in E(G)$ are distinct.. They proved that $K_2 \cup \overline{K_n}$ is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if $q \leq 1$ and every graph other than K_2 and $K_2 \cup \overline{K_n}$ are product edge-antimagic.

K. Thirusangu, E. Bala and K. Balasangu [9] introduced two new labelings called *product antimagic labeling* and *total product antimagic labeling* for *directed graphs* and showed the existence of the same for Cayley digraphs of 2-generated 2- groups. They defined a (p, q)-digraph $G(V, E)$ is $(0, 1)$ *product antimagic* if there exist a bijective function f from $E(G)$ onto the set $\{1, 2, . . . , q\}$ such that for any pair of distinct vertices

$u_i, v_j \in V(G)$, the product of the labels of the outgoing edges of v_i is distinct from the product of the labels of the outgoing edges of v_j.

In the following section we consider $V(P_n) = \{u_1, u_2, ..., u_n\}$ and $E(P_n) = \{u_1u_2, u_2u_3, ..., u_{n-1}u_n\}$ as the set of vertices and edges of the path P_n respectively. Also for the cycle C_n, $V(C_n) = \{u_1, u_2, ..., u_n\}$ and $E(C_n) = \{u_1u_2, u_2u_3, ..., u_{n-1}u_n, u_nu_1\}$ as the set of vertices and edges respectively.

2. Product Edge-Antimagic Vertex (PEAV) Labeling

Definition 2.1. We define a *product edge-antimagic vertex (PEAV) labeling* of a graph G as a bijection f from $V(G)$ to $\{1, 2, . . . , p\}$ such that, for any edge $uv \in G$, the product $f(u).f(v)$ are distinct.

Theorem 2.2. *Every path P_n, $n \geq 3$ has a PEAV labeling.*

Proof:

Let us define a vertex labeling $f: V(P_n) \to \{1, 2, ..., n\}$ as

$$f(u_i) = i \text{ where } 1 \leq i \leq n.$$

Let π_i denotes the product of the labels on the end vertices of the edge u_iu_{i+1} for $1 \leq i \leq n-1$.

Then $\pi_i = f(u_i).f(u_{i+1}) = i(i+1)$ for $1 \leq i \leq n-1$.

We observe that π_i is even and $\pi_i < \pi_j$ for $1 \leq i < j \leq n-1$.

Thus, $\pi_i \neq \pi_j$ for $1 \leq i < j \leq n-1$.

Hence f is a product edge-antimagic vertex labeling of P_n.

Theorem 2.3. *Every cycle C_n, $n \geq 3$ has a PEAV labeling.*

Proof:

Let π_i denotes the product of the labels on the end vertices of the edge u_iu_{i+1} for $1 \leq i \leq n-1$ and π_n denotes the product of the labels on the end vertices of the edge u_nu_1.

Case 1: Assume that n is odd.

Let us define a vertex labeling $f: V(C_n) \to \{1, 2, ..., n\}$ as

$$f(u_i) = i \text{ where } 1 \leq i \leq n.$$

Then $\pi_i = f(u_i).f(u_{i+1}) = i(i+1)$ for $1 \leq i \leq n-1$

and $\pi_n = f(u_n).f(u_1) = n$.

We observe that π_i is even and $\pi_i < \pi_j$ for $1 \leq i < j \leq n-1$.

Thus, $\pi_i \neq \pi_j$ for $1 \leq i < j \leq n$.

Case 2: Assume that n is even.

Let us define a vertex labeling $f: V(C_n) \to \{1, 2, ..., n\}$ as

$$f(u_i) = \begin{cases} \dfrac{i+1}{2} & ; \text{if } i \text{ is odd} \\ \dfrac{n+i}{2} & ; \text{if } i \text{ is even} \end{cases}$$

Then, for $1 \le i \le n-1$,

$$\pi_i = \begin{cases} \left(\dfrac{i+1}{2}\right)\left(\dfrac{n+i+1}{2}\right) & ; \text{if } i \text{ is odd} \\ \left(\dfrac{n+i}{2}\right)\left(\dfrac{i+2}{2}\right) & ; \text{if } i \text{ is even} \end{cases}$$

And $\pi_n = f(u_n).f(u_1) = n$.

We observe that $\pi_i < \pi_j$ for $1 \le i < j \le n-1$.

To complete the proof, we need to show that $\pi_i \ne n$ for $1 \le i \le n-1$.

Here it is clear that

$$\pi_1 = \left(\frac{1+1}{2}\right)\left(\frac{n+1+1}{2}\right) = \frac{n}{2}+1 \ne n \qquad [\because n \ge 3]$$

$$\pi_2 = \left(\frac{n+2}{2}\right)\left(\frac{2+2}{2}\right) = n+2 \ne n$$

$$\pi_3 = \left(\frac{3+1}{2}\right)\left(\frac{n+3+1}{2}\right) = n+4 \ne n$$

For $4 \le i \le n-1$, suppose $\pi_i = n$.

Then, we observe that

For odd *i*,

$$\pi_i = n \quad \Rightarrow \left(\frac{i+1}{2}\right)\left(\frac{n+i+1}{2}\right) = n$$

$$\Rightarrow (i+1)(n+i+1) = 4n$$

$$\Rightarrow (i+1)(n+i+1) = 4n$$

$$\Rightarrow n = \frac{(i+1)^2}{3-i} < 0 \text{, which is not possible}$$

Also, For even *i*,

$$\pi_i = n \quad \Rightarrow \left(\frac{n+i}{2}\right)\left(\frac{i+2}{2}\right) = n$$

$$\Rightarrow (n+i)(i+2) = 4n$$

$$\Rightarrow n = \frac{i(i+2)}{2-i} < 0 \text{, which is not possible}$$

Thus, $\pi_i \ne \pi_j$ for $1 \le i < j \le n$.

Hence *f* is a product edge antimagic vertex labeling of C_n.

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