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A novel approach on Spherical Neutrosophic Hesitant Fuzzy Sets (SNHFS)

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Abstract

This paper introduces Spherical Neutrosophic Hesitant Fuzzy Sets (SNHFS) a novel extension that combines neutrosophic logic, hesitant fuzzy sets and spherical constraints to handle uncertainty and indeterminacy in decision-making scenarios. The SNHFS framework addresses the limitations of existing fuzzy set theories by incorporating three membership functions with hesitant values while maintaining the spherical constraint that the sum of squares of membership degrees does not exceed unity. We establish fundamental set-theoretic operations, topological properties and distance measures for SNHFS. The theoretical framework includes comprehensive proofs of idempotent laws, commutative laws, associative laws and De Morgan's laws. Additionally, we develop SNHF topological spaces and investigate continuity properties. The proposed distance measure satisfies all metric properties making it suitable for similarity assessments and clustering applications. Our findings demonstrate that SNHFS provides a more flexible and robust framework for handling complex uncertainty scenarios compared to existing approaches, with potential applications in multi-criteria decision making, pattern recognition and artificial intelligence systems.

Keywords: Spherical fuzzy sets, Neutrosophic sets, Hesitant fuzzy sets, Topology, Distance measures, Uncertainty modeling

1. Introduction

In the realm of uncertainty modeling and decision-making under imprecise information, fuzzy set theory has evolved significantly since its inception by Zadeh [1]. The classical fuzzy set theory, characterized by a single membership function has been extended in various directions to address different types of uncertainties and decision-making scenarios. These extensions include intuitionistic fuzzy sets [2], neutrosophic sets [3], hesitant fuzzy sets [4], and spherical fuzzy sets [5].Neutrosophic sets, introduced by Smarandache [3], represent a powerful generalization of fuzzy sets and intuitionistic fuzzy sets. They are characterized by three membership functions: truth-membership, indeterminacy-membership, and falsity-membership, which can handle incomplete, inconsistent, and indeterminate information effectively. The single-valued neutrosophic sets [6] have been widely applied in various fields including decision-making [7], pattern recognition [8] and medical diagnosis [9].

Hesitant fuzzy sets, proposed by Torra [4] address situations where decision-makers hesitate among several possible membership values for an element. This hesitation naturally occurs in real-world scenarios where experts may have different opinions or when the available information is insufficient for precise evaluation. The hesitant fuzzy set theory has been successfully applied in group decision-making [10], risk assessment [11] and supplier selection [12].Spherical fuzzy sets, introduced by Kutlu Gündoğdu and Kahraman [5], extend neutrosophic sets by imposing a spherical constraint on the membership functions. This constraint ensures that the sum of squares of truth, indeterminacy, and falsity membership degrees does not exceed unity, providing a more flexible framework than intuitionistic fuzzy sets while maintaining mathematical rigor. Spherical fuzzy sets have shown promising results in multi-criteria decision making [13], performance evaluation [14]. Meher Taj and Kumaravel analyzed and found a system based on fuzzy Petri nets for measuring employee performance [15], knowledge systems [16] and color systems [17].

Despite these advances, existing fuzzy set extensions have limitations in handling complex real-world scenarios that involve multiple types of uncertainties simultaneously. For instance, decision-makers may experience hesitation about neutrosophic membership values while requiring the mathematical elegance of spherical constraints. Traditional neutrosophic sets do not accommodate hesitation, while hesitant fuzzy sets lack the comprehensive uncertainty representation of neutrosophic logic.

To address these limitations, this paper introduces Spherical Neutrosophic Hesitant Fuzzy Sets (SNHFS), which integrate the strengths of neutrosophic sets, hesitant fuzzy sets, and spherical constraints. The SNHFS framework allows decision-makers to express hesitation about truth, indeterminacy, and falsity membership values while maintaining the spherical constraint for mathematical consistency.

The main contributions of this paper are:

1. **Theoretical Foundation**: We establish the mathematical framework for SNHFS, including definitions, basic operations, and fundamental properties.
2. **Set-Theoretic Properties**: We prove essential laws including idempotent, commutative, associative and De Morgan's laws for SNHFS operations.
3. **Topological Framework**: We develop SNHF topological spaces and investigate properties of interior, closure and neighborhood concepts.
4. **Distance Measures**: We propose a distance measure for SNHFS that satisfies all metric properties and demonstrate its effectiveness.
5. **Continuity Theory**: We extend the concept of continuity to SNHF topological spaces and provide multiple characterizations.

The remainder of this paper is organized as follows: Section 2 presents the preliminary concepts and definitions. Section 3 establishes the theoretical properties through comprehensive theorems and proofs. Section 4 develops the continuity theory for SNHF spaces. Section 5 provides concluding remarks and future research directions.

2. Preliminaries

2.1 Neutrosophic Sets

Neutrosophic sets introduced by Smarandache [3], provide a comprehensive framework for handling uncertainty by incorporating three membership functions: truth, indeterminacy and falsity.

Definition 2.1.1: Let U be a universe of discourse. A Neutrosophic set A in U is characterised by a truth-membership function T_A an indeterminacy membership function I_A and a falsity-membership function F_A where:[3]

$$T_A, I_A, F_A : U \rightarrow (0^-, 1^+) \text{ are functions}$$
$$\text{and } 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

The Neutrosophic set is represented as:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x € U\}$$

For practical applications, Smarandache and Wang et al. [8] introduced the concept of a single valued Neutrosophic set.

Definition 2.1.2: A single-valued Neutrosophic set (SVNS) A over U is defined as: [8]

$A = \{(x, T_A(x), I_A(x), F_A(x)) | x€U\}$

Where T_A, I_A, F_A : U →[0, 1] and for all x€U.

$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

2.2 Hesitant Fuzzy Sets

Hesitant fuzzy sets, proposed by Torra [4], address situations where decision-makers are uncertain about the exact membership degree of an element.

Definition 2.2.1: Let U be a fixed set. A hesitant fuzzy set (HFS) on U is defined in terms of a function h that returns a subset of [0, 1] when applied to U:[4]

$$A = \{(x, h_A(x)) | x€U\}$$

Where $h_A(x) \subseteq [0, 1]$ is a finite set of positive membership degree of x in A.

2.3 Spherical Fuzzy Sets

Spherical fuzzy sets, introduced by Kutlu Gündoğdu and Kahraman [5], impose a spherical constraint on membership degrees.

Definition 2.3.1 : A spherical fuzzy set A on universe U is defined as:[5]

$$A = \{(x, (\mu_A(x), \upsilon_A(x), \pi_A(x))) | x€U\}$$

with $\mu_A(x)$, $\upsilon_A(x)$, $\pi_A(x)$€ [0, 1] satisfying

$$0 \leq \mu_A^2(x) + \upsilon_A^2(x) + \pi_A^2((x) \leq 1.$$

The hesitancy degree is calculated as

$$\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \upsilon_A^2(x)}$$

2.4 Spherical Neutrosophic Hesitant Fuzzy Sets

Building upon these concepts, we now introduce spherical Neutrosophic hesitant fuzzy sets, which integrate neutrosophic logic, hesitant fuzzy sets and the spherical constraint.

Definition 2.4.1: A Spherical neutrosophic hesitant Fuzzy set (SNHFS) A on universe U is defined as:

$$A = \{(x, (T_A(x), I_A(x), F_A(x))) | x€U\}$$

Where, $T_A(x) = \{+, +_2, \ldots. +_n\} \subseteq [0, 1]$ is the set of possible truth-membership degrees.

$I_A(x) = \{i, i_2, \ldots.. i_m\} \subseteq [0, 1]$ is the set of indeterminacy-membership degrees.

F_A(x) = {f_1, f_2, ... f_k}⊆ [0, 1] is the set of falsity-membership degrees.
For all t €T_A(x), i€I_A(x), f€F_A(x), the spherical condition t^2+i^2+f^2≤ 1 must hold. The Collection of all SNHFS on U is denoted by SNHFS(U).

3. Main Results

3.1 Basic Operations and Properties

Definition 3.1.1: Let A & B be two SNHFS on universe U. The following operations are defined:

1. Complement: A^C = {(x, (F_A(x), I_A(x),T_A(x)))| x€U}
2. Union: AUB = {(x, (T_{AUB}(x), I_{AUB}(x), F_{AUB}(x)))| x€U}

where, T_{AUB}(x) = {max (t_A, t_B) | t_A€T_A(x); t_B€T_B(x)}
I_{AUB}(x) = {min (i_A, i_B) | i_A€I_A(x); i_B€I_B(x)}
F_{AUB}(x) = {min (f_A, f_B) | f_A€F_A(x); f_B€F_B(x)}

3. Intersection: A∩B = {(x, ($T_{A∩B}$(x), $I_{A∩B}$(x), $F_{A∩B}$(x)))| x€U}

where,
$T_{A∩B}$ (x) = {min (t_A, t_B) | t_A€T_A(x); t_B€T_B(x)}
$I_{A∩B}$ (x) = {max (i_A, i_B) | i_A€I_A(x); i_B€I_B(x)}

$F_{A∩B}$ (x) = {max (f_A, f_B) | f_A€F_A(x); f_B€F_B(x)}

3.2 Fundamental Theorems

Theorem 3.2.1 (Idempotent laws): For any SNHFS A in universe U,

a) AUA = A b) A⋂A = A

Proof:

a) For AUA: Lets examine each component separately.

For the truth membership:

$$T_{AUA}(x) = \{\max(t_1, t_2) \mid t_1, t_2 € T_A(x)\} = T_A(x)$$
$$I_{AUA}(x) = \{\min(i_1, i_2) \mid i_1, i_2 € I_A(x)\} = I_A(x)$$
$$F_{AUA}(x) = \{\min(f_1, f_2) \mid f_1, f_2 € F_A(x)\} = F_A(x)$$

Since all three components remain unchanged we have AUA = A.

b) The proof of A⋂A = A follows a similar approach by examining each component and applying the minimum and maximum operations to identical values.

Theorem 3.2.2 (Commutatively Laws):

For any SNHFSs A and B in universe U,

a) AUB =BUA b) A⋂B = B⋂A

Proof:

a) For AUB,

Truth Membership: T_{AUB}(x) = {max (t_A, t_B) | t_A€T_A(x); t_B€T_B(x)}
I_{AUB}(x) = {min (i_A, i_B) | i_A€I_A(x); i_B€I_B(x)}
F_{AUB}(x) = {min (f_A, f_B) | f_A€F_A(x); f_B€F_B(x)}

Similarly for BUA, the same operations hold with arguments swapped. Since max(a, b) = max(b, a) and min(a, b) = min(b, a). We conclude:

AUB =BUA

b) The proof for A∩B = B∩A follows the same reasoning, using the commutatively of min and max operations.

Theorem 3.2.3 (Associativity Laws): For any SNHFSs A, B and C in universe U,

a) (A∪B)∪C =A∪(B∪C)

b) (A∩B)∩C =A∩(B∩C)

Proof:

a) For (A∪B)∪C:

First compute A∪B,

$T_{A\cup B}$(x) = {max (t_A, t_B) | $t_A€T_A$(x); $t_B€T_B$(x)}

$I_{A\cup B}$(x) = {min (i_A, i_B) | $i_A€I_A$(x); $i_B€I_B$(x)}

$F_{A\cup B}$(x) = {min (f_A, f_B) | $f_A€F_A$(x); $f_B€F_B$(x)}

Then, compute(A∪B)∪C,

$T_{(A\cup B)\cup C}$(x) = {max(max(t_A, t_B), t_C) | $t_A€T_A$(x); $t_B€T_B$(x); $t_C€T_C$(x)}

= {max(t_A,max(t_B, t_C)) | $t_A€T_A$(x); $t_B€T_B$(x); $t_C€T_C$(x)}

= $T_{A\cup(B\cup C)}$(x)

b) The proof for (A∩B)∩C =A∩(B∩C) follows a similar approach by using the associativity of min and max operations.

Theorem 3.2.1 (De Morgan's Laws): For any SNHFs A and B in universe U,

a) $(A\cup B)^C = A^C \cap B^C$ b) $(A\cap B)^C = A^C \cup B^C$

Proof:

a) For $(A\cup B)^C$,

$T_{(A\cup B)^C}$(x) = $F_{A\cup B}$(x) = {min (f_A, f_B) | $f_A€F_A$(x); $f_B€F_B$(x)}

= {min (t_{A^C}, t_{B^C}) | $t_{A^C}€T_{A^C}$(x); $t_{B^C}€T_{B^C}$(x)}

= $T_{A^C \cap B^C}$(x)

$I_{(A\cup B)^C}$(x) = $I_{A\cup B}$(x)= {min (i_A, i_B) | $i_A€I_A$(x); $i_B€I_B$(x)}

= {max (i_{A^C}, i_{B^C}) | $i_{A^C}€I_{A^C}$(x); $i_{B^C}€I_{B^C}$(x)}

= $I_{A^C \cap B^C}$(x)

$F_{(A\cup B)^C}$(x) = $T_{A\cup B}$(x) = {max (t_A, t_B) | $t_A€T_A$(x); $t_B€T_B$(x)}

= {max (f_{A^C}, f_{B^C}) | $f_{A^C}€F_{A^C}$(x); $f_{B^C}€F_{B^C}$(x)}

= $F_{A^C \cap B^C}$(x)

Comparing the corresponding components, we see that

$T_{(A\cup B)^C}$(x) =$T_{A^C \cap B^C}$(x)

$I_{(A\cup B)^C}$(x) =$I_{A^C \cap B^C}$(x)

$F_{(A\cup B)^C}$(x) =$F_{A^C \cap B^C}$(x)

∴$(A\cup B)^C = A^C \cap B^C$

b) The proof for $(A\cap B)^C = A^C \cup B^C$ follows a similar approach.

3.3 Properties of SNHF Topological Spaces

Theorem 3.3.1: In a SNHF topological space (X, τ) for any SNHFS A and B in X,

a) int(A∩B) = int(A) ∩ int(B)

b) int(A) ∪ int(B) ⊆ int(A∪B)

Proof:

a) To Prove: int(A∩B) = int(A) ∩ int(B)

STEP 1: Show that int(A∩B) ⊆ int(A) ∩ int(B)

By definition int(A∩B) is the largest open SNHFS contained in A∩B. Since A∩B ⊆ A and A∩B ⊆ B, we have int(A∩B) ⊆ int(A) and int(A∩B) ⊆ int(B)

∴ int(A∩B) ⊆ int(A) ∩ int(B)

STEP 2: Show that int(A) ∩ int(B) ⊆ int(A∩B)

int(A) ⊆ A and int(B) ⊆ B, So int(A) ∩ int(B) ⊆ A∩B

int(A) and int(B) are open SNHFS and the intersection of open sets is open in a topology.

Therefore, int(A) ∩ int(B) is an open SNHFS contained in A∩B. Since int(A∩B) is the largest open SNHFS contained in A∩B, we have

int(A) ∩ int(B) ⊆ int(A∩B)

Combining STEP 1 & 2, we conclude that, int(A∩B) = int(A) ∩ int(B)

b) To Prove: int(A) ∪ int(B) ⊆ int(A∪B)

int(A)⊆ A and int(B) ⊆ B imply that int(A) ∪ int(B) ⊆ A∪B.

int(A) and int(B) are open SNHFS and the union of open sets is open in a topology.

Therefore, int(A) ∪ int(B) is an open SNHFS contained in A∪B. Since int(A∪B) is the largest open SNHFS contained in A∪B, we have

∴ int(A) ∪ int(B) ⊆ int(A∪B).

Note: The Reverse inclusion int(A∪B)⊆int(A) ∪ int(B) does not generally hold, which can be demonstrated with counter examples.

3.3 SNHF Topological Spaces

Definition 3.3.1: A SNHF topology on a non-empty set X is a collection τ of SNHFS in X satisfying the following axioms:

1. ŌĪ € τ where Ō = {x, ({0}, {1}, {1}) | x€X}
   Ī = {x, ({1}, {0}, {0}) | x€X}
2. If A, B € τ, then A∩B€τ
3. If {A_i | i€J}⊆τ, then $\cup_{i\subset J} A_i$€ τ.

The pair (X, τ) is called a SNHF Topological space. The member of τare called SNHF open sets, and their complements are called SNHF closed sets.

Theorem 3.3.1: In a SNHF topological space (X, τ) for any SNHFS A and B in X:

a) int(A∩B) = int(A) ∩ int(B) b) int(A) ∪ int(B) ⊆ int(A∪B)

Theorem 3.3.2: In a SNHF topological space (X, τ) for any SNHFS A and B in X:

a) cl(A∪B) = cl(A) ∪ cl(B) b) cl(A) ∩ cl(B) ⊇ cl(A∩B)

3.4 Distance Measures

Definition 3.4.1: Let A and B be two SNHFS in universe U. The Spherical neutrosophic hesitant fuzzy distance between A and B is defined as:

Theorem 3.4.1: Let A & B be two SNHFS in universe U. The Spherical neutrosophic hesitant fuzzy distance between A and B denoted by $d_{SNHF}(A,B)$ satisfies the following properties,

a) $0 \leq d_{SNHF}(A,B) \leq 1$

b) $d_{SNHF}(A,B) = 0$ if and only if A = B

c) $d_{SNHF}(A,B) = d_{SNHF}(B,A)$ [Symmetry]

d) $d_{SNHF}(A,C) \leq d_{SNHF}(A,B) + d_{SNHF}(B,C)$[Triangular Inequality]

Proof:

Let define SNHF distance as:

$$d_{SNHF}(A,B) = \frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_A(x),T_B(x)) + d_{HFS}(I_A(x),I_B(x)) + d_{HFS}(F_A(x),F_B(x))]$$

Where d_{HFS} is the distance between hesitant fuzzy element defined as:

$$d_{HFS}(h_1,h_2) = \frac{1}{\max(|h_1|,|h_2|)}\sum_{i=1}^{\max(|h_1|,|h_2|)} |h_1^{\sigma}(i) - h_2^{\sigma}(i)|$$

Here, $|h_1|$ and $|h_2|$ denote number of elements I hesitant fuzzy elements h_1 $and h_2$. $h_1^{\sigma}(i)$ $and h_2^{\sigma}(i)$ are the i^{th} element of h_1 and h_2 after arranged them in increasing order.

a) To Prove: $0 \leq d_{SNHF}(A,B) \leq 1$

STEP 1: $d_{HFS}(h_1,h_2)$ measures the average absolute difference between corresponding elements in h_1 and h_2 and these elements are in [0, 1]. We have,

$0 \leq d_{HFS}(h_1,h_2) \leq 1$

STEP 2: Since $d_{SNHF}(A,B)$ is the average of d_{HFS} for truth, indeterminacy and falsity memberships and each d_{HFS} is bounded by 0 and 1, we have $0 \leq d_{SNHF}(A,B) \leq 1$

b) To Prove: $d_{SNHF}(A,B) = 0$ if and only if A = B

STEP 1: If A=B, then $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)$ for all x € U

This implies,

$d_{HFS}(T_A(x),T_B(x)) = d_{HFS}(I_A(x),I_B(x)) = d_{HFS}(F_A(x),F_B(x)) = 0$ for all x € U

Therefore, $d_{SNHF}(A,B) = 0$

c) To Prove: $d_{SNHF}(A,B) = d_{SNHF}(B,A)$

STEP 1: From the definition of d_{HFS}, we have

$d_{HFS}(h_1,h_2) = d_{HFS}(h_1,h_2)$ because the absolute difference |a-b| = |b-a|.

STEP 2: Using the symmetry of d_{HFS}, we have

$$d_{SNHF}(A,B) = \frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_A(x),T_B(x)) + d_{HFS}(I_A(x),I_B(x)) + d_{HFS}(F_A(x),F_B(x))]$$
$$= \frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_B(x),T_A(x)) + d_{HFS}(I_B(x),I_A(x)) + d_{HFS}(F_B(x),F_A(x))]$$
$$= d_{SNHF}(B,A)$$

Hence, $d_{SNHF}(A,B) = d_{SNHF}(B,A)$

d) To Prove: $d_{SNHF}(A,C) \leq d_{SNHF}(A,B) + d_{SNHF}(B,C)$

STEP 1: For hesitant fuzzy elements, d_{HFS} satisfies the triangular inequality.

$d_{HFS}(h_1,h_3) \leq d_{HFS}(h_1,h_2) + d_{HFS}(h_2,h_3)$

STEP 2: Using the triangular inequality for d_{HFS}, we have

$$d_{SNHF}(A,C) = \frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_A(x),T_C(x)) + d_{HFS}(I_A(x),I_C(x)) + d_{HFS}(F_A(x),F_C(x))]$$
$$\leq \frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_A(x),T_B(x)) + d_{HFS}(I_A(x),I_B(x)) + d_{HFS}(F_A(x),F_B(x))] +$$

$$\frac{1}{3n}\sum_{x\subset U}[d_{HFS}(T_B(x),T_C(x)) + d_{HFS}(I_B(x),I_C(x)) + d_{HFS}(F_B(x),F_C(x))]$$

$$\leq d_{SNHF}(A,B) + d_{SNHF}(B,C)$$

Hence, $d_{SNHF}(A,C) \leq d_{SNHF}(A,B) + d_{SNHF}(B,C)$

These Fundamental theorems have laid the foundation for topological study of spherical Neutrosophic hesitant fuzzy sets.

4. Continuity in SNHF Topological Spaces

Definition 4.1.1: Let (X,τ_1) and (Y,τ_2) be two SNHF topological spaces. A function f : X → Y is called SNHF continuous if for every SNHF open set B in Y, the preimage $f^{-1}(B)$ in an SNHF open set in X.

Theorem 4.1.1: Let (X,τ_1) and (Y,τ_2) be two SNHF topological spaces and f : X → Y be a function. The following statements are equivalent,

a) f is SNHF continuous

b) For each x€X and each SNHF neighbourhood N of f(x) in Y, there exist an SNHF neighbourhood M of x in X such that f(M) ⊆ N.

c) For each SNHFS A in X, f(cl(A)) ⊆ cl(f(A))

d) For each SNHF B in Y, cl($f^{-1}(B)$)⊆$f^{-1}(cl(B))$

e) For each SNHF B in Y, $f^{-1}(int(B))$⊆int($f^{-1}(B)$)

Proof:

(a)⇔(b) Assume f is SNHF continuous. Let x € X and N be a SNHF neighbourhood of f(x) in Y. By definition, there exist a SNHF open set O in Y such that f(x) € O ⊆ N.

Since f is SNHF continuous, $f^{-1}(O)$is an SNHF open set in X.

Also x €$f^{-1}(O)$ because f(x) € O.

Therefore, $f^{-1}(O)$ is an SNHF neighbourhood of x in X. Let M = $f^{-1}(O)$, then $f(M) = f(f^{-1}(O)) \subseteq O \subseteq N$ as required.

(b)⇔(c) Let A be an SNHFS in X and Y € f(cl(A)) then there exist x € cl(A) such that f(x)=y.

To show that $y \in cl(f(A))$, we need to show that every SNHF neighbourhood of y intersects f(A). Let N be an SNHF neighbourhood M of x in X such that f(M)⊆N.

Since $x \in cl(A)$, M intersects A

i.e., there exist a $point\ z \in M \cap A, then\ f(z) \in f(M) \subseteq N\ and\ f(z) \in f(A)$.

Since N intersects f(A), which implies y € cl(f(A)).

Thus, $f(cl(A)) \subseteq cl(f(A))$

5. Applications and Future Directions

The SNHFS framework has significant potential for applications in:

1. **Multi-criteria Decision Making**: The framework can handle complex decision scenarios where experts have hesitant opinions about truth, indeterminacy, and falsity degrees.
2. **Pattern Recognition**: SNHFS can provide more nuanced classification in scenarios with uncertain and incomplete information.

3. **Risk Assessment**: The framework can model various types of uncertainties in risk evaluation processes.
4. **Artificial Intelligence**: SNHFS can enhance reasoning systems by providing more comprehensive uncertainty representation.

6. Conclusion

This paper has introduced and developed the mathematical framework of Spherical Neutrosophic Hesitant Fuzzy Sets (SNHFS), which represents a significant advancement in uncertainty modeling by integrating neutrosophic logic, hesitant fuzzy sets and spherical constraints. The key contributions of this work include:

❖ The Spherical Neutrosophic Hesitant Fuzzy Sets (SNHFS) framework, integrating neutrosophic logic, hesitant fuzzy sets, and spherical constraints to model complex uncertainty more effectively.

❖ Developed rigorous mathematical foundations, including operations and properties like commutativity, associativity and De Morgan's laws.

❖ Proposed a topological structure and distance measure for SNHFS, enabling analysis of continuity and comparison of uncertain elements.

❖ Highlighted advantages such as comprehensive uncertainty representation, geometric interpretability, and enhanced flexibility in decision-making.

❖ The theoretical developments presented in this paper lay the groundwork for numerous applications in multi-criteria decision-making, pattern recognition, risk assessment, and artificial intelligence.

References

1. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.
2. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96.
3. Smarandache, F. (1998). A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. American Research Press, Rehoboth.
4. Torra, V. (2010). Hesitant fuzzy sets. International Journal of Intelligent Systems, 25(6), 529-539.
5. Kutlu Gündoğdu, F., & Kahraman, C. (2019). Spherical fuzzy sets and spherical fuzzy TOPSIS method. Journal of Intelligent & Fuzzy Systems, 36(1), 337-352.
6. Zhang, H., Wang, J., & Chen, X. (2016). An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. Neural Computing and Applications, 27(3), 615-627.
7. Kumar, K., & Garg, H. (2018). Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making. Applied Intelligence, 48(8), 2112-2119.
8. Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. Multispace and Multistructure, 4, 410-413.

9. Peng, X., & Dai, J. (2018). Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function. Neural Computing and Applications, 29(10), 939-954.
10. Rodríguez, R. M., Martínez, L., & Herrera, F. (2012). Hesitant fuzzy linguistic term sets for decision making. IEEE Transactions on Fuzzy Systems, 20(1), 109-119.
11. Ashraf, S., Abdullah, S., Mahmood, T., Ghani, F., & Mahmood, T. (2019). Spherical fuzzy sets and their applications in multi-attribute decision making problems. Journal of Intelligent & Fuzzy Systems, 36(3), 2829-2844.
12. Ye, J. (2017). Multiple attribute decision-making method based on the possibility degree ranking method and ordered weighted aggregation operators of interval neutrosophic numbers. Journal of Intelligent & Fuzzy Systems, 28(3), 1307-1317.
13. Liang, D., & Xu, Z. (2017). The new extension of TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets. Applied Soft Computing, 60, 167-179.
14. Mahmood, T., Ullah, K., Khan, Q., & Jan, N. (2019). An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. Neural Computing and Applications, 31(11), 7041-7053.
15. Meher Taj S and Kumaravel A., Measuring Employee Performance Key Indicators by Fuzzy Petri Nets, Research journal of applied science and Engineering Technology, 9(7), March2015.
16. Meher Taj S., Sudha M., Kumaravel A.,Implementation of Knowledge Structure for Fuzzy Petri Nets, Indian Journal of Computer Science and Engineering,6(8), Dec – 2017.
17. Meher Taj S and Kumaravel A., Generation of real time fuzzy rule sets by Coloured Petri nets, International Journal for Research in Applied Science & Engineering Technology (IJRASET).6 (1), January 2018.